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Effects of weak nonlinearity on dispersion relation and frequency band-gaps of a periodic Bernoulli-Euler beam

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Abstract

The paper deals with analytically predicting the effects of weak nonlinearity on the dispersion relation and frequency band-gaps of a periodic Bernoulli-Euler beam performing bending oscillations. Two cases are considered: 1) large transverse deflections, where nonlinear (true) curvature, nonlinear material, and nonlinear inertia due to longitudinal motions of the beam are taken into account, and 2) mid-plane stretching nonlinearity. A novel approach is employed, the Method of Varying Amplitudes. As a result the isolated as well as combined effects of the considered sources of nonlinearities are revealed. It is shown that nonlinear inertia has the most substantial impact on the dispersion relation of a non-uniform beam. It appears to remove all band-gaps, by making the wave motion either strongly nonlinear (for very small beam deflections) or weakly nonlinear and featuring no band-gaps (for larger beam deflections). Explanations of the revealed effects are suggested, and validated by numerical simulation.

Keywords: elastic wave propagation; dispersion relation; frequency band-gaps; weak nonlinearity; periodic Bernoulli-Euler beam; method of varying amplitudes.

1. Introduction

The analysis of the behaviour of linear periodic structures can be traced back over 300 years to Sir Isaac Newton [1], but until Rayleigh's work [2] the systems considered were lumped masses joined by massless springs. Much attention was given to this topic in 20[th] century, the classical works [1, 3, 4] can be mentioned here. In recent years the topic has experienced rising interest, e.g. [5-9]. An essential feature of periodic structures is the presence of frequency band-gaps, i.e. frequency ranges in which waves cannot propagate. Determination of band-gaps and the corresponding attenuation levels is an important practical problem [3-9]. A large variety of analytical methods was developed for solving it, most of them based on Floquet theory [1]; e.g. this holds for the classical Hill's method of infinite determinants [10,11], and the method of space-harmonics [12]. However, application of these for nonlinear problems is impossible or cumbersome, since Floquet theory is applicable for linear systems only. Thus the nonlinear effects for periodic structures are not yet fully uncovered, while at the same time applications may demand effects of nonlinearity on structural response to be accounted for. Only a few papers are devoted to this topic (e.g. [13,14]), all of them considering lumped-parameter models, i.e. mass-spring chains. However, the ability of such models to properly describe structural response is considerably restricted, see, e.g. the classical works [1,12], and the more recent papers [8,9].

The present paper abandons this simplification and deals with analytically predicting dynamic responses for a nonlinear continuous elastic periodic structure. Specifically, the effects of weak nonlinearity on the dispersion relation and frequency band-gaps of a periodic Bernoulli-Euler beam performing bending oscillations are analyzed. The periodic modulation of beam structural properties is not required to be small or piecewise constant, meaning that even the corresponding linear problem does not allow an exact solution. Various sources of nonlinearity are analyzed: nonlinear (true) curvature, nonlinear inertia due to longitudinal beam motions, nonlinear material, and the nonlinearity associated with mid-plane stretching.

A novel approach is employed, the *Method of Varying Amplitudes* (MVA) [15,16]. This approach is inspired by the method of direct separation of motions (MDSM) [17,18], and may be considered a natural continuation of the classical methods of harmonic balance [11] and averaging [19-21]. It implies representing a solution in the form of a harmonic series with varying amplitudes; however, in contrast to the asymptotic methods, the amplitudes are not required to vary slowly. The approach is strongly related also to Hill's method of infinite determinants [1, 10, 11], and to the method of space-harmonics [12].

Possible sources of nonlinearities for a Bernoulli-Euler beam performing bending oscillations has been discussed in many works, see e.g. the classical monograph [11], and the paper [22]. In [11] the main sources were identified as nonlinear stiffness and nonlinear inertia. It was noted that the character of the nonlinearity strongly depends on specific boundary conditions applied to the beam. For example, when there is no restriction on longitudinal motions of beam ends, large deflections are possible, so that nonlinear (true) curvature and nonlinear inertia due to longitudinal motions of the beam should be taken into account. The effects of nonlinear material may also be of significance in this case. If both ends of the beam are restricted to move in the longitudinal direction another source of nonlinearity becomes most important, namely mid-plane stretching. This nonlinearity appears to be much stronger than the others [23,24], and influences the beam response already at relatively small deflections.

Since in real structures some of the nonlinearities are imposed due to boundary conditions, finite structures should be considered. On the other hand, the analysis of dispersion relations and frequency band-gaps usually implies considering *infinite* structures [1,12]. The transition from infinite to finite structures, and the discussion of the validity of dispersion relations and band-gaps for finite structures are given in many papers, e.g. [1,5,7,12,25]. The basic assumption in such a transition is that the considered structure is sufficiently long for waves from band-gap ranges to be strongly attenuated before reaching the boundaries.

Section 2 is concerned with the formulation of the governing equations of transverse motions of the beam and their brief analysis. In Section 3 the equations are solved by the MVA; Section 4 presents the obtained dispersion relations and reveals the effects of nonlinearities on the frequency band-gaps. Section 5 is concerned with the discussion and numerical validation of the results.

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2. Governing equations

2.1. Case A: Beam unrestricted longitudinally; large deflections possible

Consider the case with no restriction on longitudinal motions of the beam, and large transverse deflections possible. The internal bending moment of a Bernoulli-Euler beam with spatially varying properties is defined by:

$$M(\tilde{x},\tilde{t}) = EI(\tilde{x})\kappa\,, \tag{1}$$

where I is the moment of inertia of the cross-section, E Young's modulus of the beam material, $\tilde{x}$ the axial coordinate along the deformed beam, and $\kappa(\tilde{x},\tilde{t})$ the nonlinear (true) curvature [11, 22]:

$$\kappa(\tilde{x},\tilde{t}) = \frac{\tilde{w}''}{\sqrt{1-(\tilde{w}')^2}} \approx \left(1+\frac{1}{2}(\tilde{w}')^2\right)\tilde{w}'' \tag{2}$$

Here $\tilde{w}(\tilde{x},\tilde{t})$ is the transverse displacement of the beam at time $\tilde{t}$ and axial coordinate $\tilde{x}$, primes denote derivatives with respect to $\tilde{x}$, and the approximation in (2) assumes $(\tilde{w}')^2 << 1$. Taking into account effects of nonlinear material [11] we rewrite (1) as:

$$M(\tilde{x},\tilde{t}) = EI(\tilde{x})\left[\kappa - \tilde{\beta}_n\kappa^3\right], \tag{3}$$

where the coefficient $\tilde{\beta}_n$ defines the nonlinearity of the beam material stress-strain relation; note that for most materials the nonlinearity is symmetric and of "softening" type, i.e. there is no quadratic term, and $\tilde{\beta}_n > 0$ [11]. Inserting (2) into (3) and keeping nonlinearities to third order, we obtain:

$$M(\tilde{x},\tilde{t}) = EI(\tilde{x})\left[1+\frac{1}{2}\left(\tilde{w}'\right)^2 - \tilde{\beta}_n\left(\tilde{w}''\right)^2\right]\tilde{w}''\,, \tag{4}$$

The longitudinal displacement of the beam cross-section due to large transverse deflections is [11,22]:

$$u(\tilde{x},\tilde{t}) = -\frac{1}{2}\int\left(\tilde{w}'\right)^2 d\tilde{x}\,. \tag{5}$$

With no restriction on longitudinal boundary motions, Newton's second law applied in the $\tilde{x}$-direction gives:

$$\tilde{N}' = \rho A(\tilde{x})\frac{\partial^2 u}{\partial \tilde{t}^2}\,, \tag{6}$$

where $\tilde{N}$ is the additional longitudinal force due to effects of inertia, and $\rho A(\tilde{x})$ is the beam mass per unit length. By integration and with (5) one obtains:

$$\tilde{N}(\tilde{x},\tilde{t}) = \int \rho A(\tilde{x})\frac{\partial^2 u}{\partial \tilde{t}^2}d\tilde{x} = -\frac{1}{2}\int \rho A(\tilde{x})\frac{\partial^2}{\partial \tilde{t}^2}\left[\int\left(\tilde{w}'\right)^2 d\tilde{x}\right]d\tilde{x}\,. \tag{7}$$

The beam mass per unit length $\rho A(\tilde{x})$ and (linear) bending stiffness $EI(\tilde{x})$ are assumed to be periodically varying in the axial coordinate $\tilde{x}$:

$$\rho A(\tilde{x}) = \rho A(\tilde{x}+\Theta)\,,\; EI(\tilde{x}) = EI(\tilde{x}+\Theta)\,, \tag{8}$$

where Θ is the period of modulation, and

$$I(\tilde{x}) = \left(r(\tilde{x})\right)^2 A(\tilde{x})\,, \tag{9}$$

where $r(\tilde{x})$ is the radius of gyration of the cross-sectional area $A(\tilde{x})$. Expanding $\rho A(\tilde{x})$ and $EI(\tilde{x})$ in a Fourier series gives:

$$\begin{aligned} \rho A(\tilde{x}) &= \rho A_0\left[1+\sum_{m=1}^{\infty}\chi_{A,m}\sin\left(\frac{2\pi}{\Theta}m\tilde{x}+\phi_{A,m}\right)\right], \\ EI(\tilde{x}) &= EI_0\left[1+\sum_{m=1}^{\infty}\chi_{I,m}\sin\left(\frac{2\pi}{\Theta}m\tilde{x}+\phi_{I,m}\right)\right]. \end{aligned} \tag{10}$$

Our aim is to reveal the principle effects of nonlinearities on the dispersion relation and frequency band-gaps of a periodic beam. Consequently, as a first approximation, only the fundamental harmonic is accounted for in (10), so that:

$$\begin{aligned} \rho A(\tilde{x}) &= \rho A_0(1+\chi_A\sin(k\tilde{x}+\phi)), \\ EI(\tilde{x}) &= EI_0(1+\chi_I\sin(k\tilde{x}+\phi)), \end{aligned} \tag{11}$$

where $0\le\chi_A<1$, $0\le\chi_I<1$, $k=2\pi/\Theta$, and in the simplest case of constant ρ, E, and r we have $\chi_A=\chi_I$. Here a more general case is considered, where the modulation amplitudes χ_A and χ_I are not required to be equal, though modulations of the beam mass per unit length and stiffness have the same phase shift ϕ. According to [1] the approximation similar to (11) is valid for predicting at least the lowest two band-gaps of a periodic structure.

The governing equation of transverse motions of the beam is [11,23]:

$$M''-\left(\tilde{N}\tilde{w}'\right)'+\rho A(\tilde{x})\frac{\partial^2\tilde{w}}{\partial\tilde{t}^2}=0, \tag{12}$$

which assumes waves much longer than the height of the beam, so that the classical Bernoulli-Euler theory holds, and rotary inertia and shear deflections can be ignored. Dissipation is not taken into account, which is typical [1,9,12] for studying dispersion relations and frequency band-gaps of periodic structures. Inserting (4) and (11) into (12) gives:

$$\rho A_0\left(1+\chi_A\sin(k\tilde{x}+\phi)\right)\frac{\partial^2\tilde{w}}{\partial\tilde{t}^2}-\left(\tilde{N}\tilde{w}'\right)'+EI_0\left[\left(1+\chi_I\sin(k\tilde{x}+\phi)\right)\left(1+\frac{1}{2}\left(\tilde{w}'\right)^2-\tilde{\beta}_n\left(\tilde{w}''\right)^2\right)\tilde{w}''\right]''=0. \tag{13}$$

Introducing non-dimensional variables $x=k\tilde{x}+\phi$, $t=\tilde{\omega}\tilde{t}$, and $w=k\tilde{w}$, where $\tilde{\omega}=k^2\sqrt{EI_0/(\rho A_0)}$ is frequency of waves with length $2\pi/k$ propagating in the corresponding uniform beam, (13) can be rewritten in dimensionless form:

$$(1+\chi_A\sin x)\frac{\partial^2 w}{\partial t^2}+\left[\left(1+\chi_I\sin x\right)\left(1+\frac{1}{2}\left(w'\right)^2-\beta_n\left(w''\right)^2\right)w''\right]''-\left(Nw'\right)'=0 \tag{14}$$

where $\beta_n=\tilde{\beta}_n k^2$,

$$N(x,t)=-\frac{1}{2}\int(1+\chi_A\sin x)\frac{\partial^2}{\partial t^2}\left[\int\left(w'\right)^2dx\right]dx, \tag{15}$$

and primes now denote derivatives with respect to the non-dimensional spatial coordinate x.

Since $w=k\tilde{w}$ the effect of the nonlinearities depends not only on magnitude of the physical deflections $\tilde{w}$, but also on the value of k, so that for large k the effect can be significant even for relatively small transverse deflections of the beam.

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Solutions of (14) are sought in the form of a series, with over-bars denoting complex conjugation:

$$w(x,t)=\varphi(x)e^{i\omega t}+\bar{\varphi}(x)e^{-i\omega t}+\varphi_n(x)e^{3i\omega t}+\bar{\varphi}_n(x)e^{-3i\omega t}+\ldots, \tag{16}$$

which is typical for problems involving oscillations of weakly nonlinear (homogeneous) structures with only symmetric forces being present [26,27]. Only waves whose period is of the same order as, or much larger than, the period Θ of modulation are considered, so high frequency oscillations are out of scope. Also, nonlinearities are assumed to be weak, permitting only the fundamental harmonic in (16) to be included. This simplification, which is validated in Section 5.3, also agrees with the low frequency applicability range of Bernoulli-Euler theory.

Substituting (16) and (15) into (14) and balancing terms at the fundamental harmonic ω, one obtains the governing ordinary differential equation for $\varphi(x)$:

$$\begin{aligned}&\left[(1+\chi_I\sin x)\left(\varphi''+\frac{1}{2}\left(2\varphi''\varphi'\bar{\varphi}'+\bar{\varphi}''(\varphi')^2\right)-3\beta_n(\varphi'')^2\bar{\varphi}''\right)\right]''-\omega^2\left[(1+\chi_A\sin x)\varphi\right.\\&\left.+2\left[\bar{\varphi}'\int\left((1+\chi_A\sin x)\int(\varphi')^2dx\right)dx\right]'\right]=0.\end{aligned} \tag{17}$$

Since only waves with a period of the same order as, or much larger than, the period Θ of modulation are considered, and due to the choice of the non-dimensional variables, we have $\omega=O(1)$ (which comprises also the case $\omega<<1$).

The integral term in (17) represents nonlinear inertia, while terms with β_n are related to nonlinear material, and the remaining nonlinear terms are due to the true measure of curvature (2). Note that though $\tilde{\beta}_n<<1$, the coefficient $\beta_n=\tilde{\beta}_n k^2$ is only small when $k=O(1)$, i.e. when Θ is not small.

2.2. Case B: Mid-plane stretching

Now consider the case when both ends of the beam are restricted to move in the longitudinal direction, and mid-plane stretching occurs. To transversely deform such a beam considerably more energy should be supplied, since bending is coupled with axial stretching of the beam. Consequently one can expect transverse deformations to be much smaller than in Case A (Section 2.1), so that the linear measure of curvature and material stress-strain relation can be adopted, and non-linear inertia can be neglected [23]. In the absence of external axial forces, the assumption regarding longitudinal inertia implies that:

$$\tilde{N}'=0, \tag{18}$$

where prime denotes derivative with respect to $\tilde{x}$; this means that for any $\tilde{t}$ the longitudinal force is constant throughout the beam. By Hooke's law [23,24]:

$$\tilde{N}(\tilde{x},\tilde{t})=EA(\tilde{x})\Lambda, \tag{19}$$

where $\Lambda(\tilde{x},\tilde{t})$ is the full axial strain:

$$\Lambda(\tilde{x},\tilde{t})=\sqrt{(1+u')^2+(\tilde{w}')^2}-1\approx u'+\frac{1}{2}(\tilde{w}')^2, \tag{20}$$

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where all variables have the same meaning as for Case A. Consequently, by (19) and (20):

$$u' = \frac{\tilde{N}}{EA(\tilde{x})} - \frac{1}{2}(\tilde{w}')^2, \tag{21}$$

so that the longitudinal displacement is

$$u(\tilde{x},\tilde{t}) = \int\left(\frac{\tilde{N}}{EA(\tilde{x})} - \frac{1}{2}(\tilde{w}')^2\right)d\tilde{x} = \tilde{N}\int\frac{1}{EA(\tilde{x})}d\tilde{x} - \frac{1}{2}\int(\tilde{w}')^2\,d\tilde{x}, \tag{22}$$

Now, by contrast to Case A, the beam ends are restricted to move longitudinally, so that $u(\tilde{l}) - u(0) = \eta\tilde{l}$, where η describes a small initial stretch of the beam, and $\tilde{l}$ is the beam length. Imposing this condition with (22) and solving for $\tilde{N}$ gives:

$$\tilde{N}(\tilde{x},\tilde{t}) = \frac{1}{\int_0^{\tilde{l}}\frac{1}{EA(\tilde{x})}d\tilde{x}}\left(\eta\tilde{l} + \frac{1}{2}\int_0^{\tilde{l}}(\tilde{w}')^2\,d\tilde{x}\right), \tag{23}$$

Inserting (23) into (12), adopting the linearized curvature, and neglecting possible effects of nonlinear material, one obtains:

$$\left(EI(\tilde{x})\tilde{w}''\right)'' + \rho A(\tilde{x})\frac{\partial^2\tilde{w}}{\partial\tilde{t}^2} - \frac{1}{\int_0^{\tilde{l}}\frac{1}{EA(\tilde{x})}d\tilde{x}}\left(\eta\tilde{l} + \frac{1}{2}\int_0^{\tilde{l}}(\tilde{w}')^2\,d\tilde{x}\right)\tilde{w}'' = 0 \tag{24}$$

Assuming the variation (11) of spatial properties, (24) becomes:

$$(1+\chi_A\sin x)\frac{\partial^2 w}{\partial t^2} + \left[(1+\chi_I\sin x)w''\right]'' - \mu\frac{1}{\int_\phi^{l+\phi}\frac{1}{1+\chi_A\sin x}dx}\left(\eta l + \frac{1}{2}\int_\phi^{l+\phi}(w')^2\,dx\right)w'' = 0, \tag{25}$$

where all parameters and variables are dimensionless and have the same meaning as in Section 2.1, $l = k\tilde{l}$ is the non-dimensional beam length, $\mu = A_0/(I_0k^2) = (kr_0)^{-2}$ where r_0 is radius of gyration of the corresponding uniform beam, and here again primes denote derivatives with respect to x.

Employing Bernoulli-Euler theory and considering waves of length much larger than the height of the beam implies that

$$\tilde{\lambda} >> r(\tilde{x}) = \sqrt{\frac{I(\tilde{x})}{A(\tilde{x})}}, \tag{26}$$

where $\tilde{\lambda}$ is wave length. Taking into account also that only waves with a period of the same order as, or much larger than, the period of property modulation are considered, i.e.

$$\tilde{\lambda}^{-1} = O(k), \tag{27}$$

one obtains that (26) is satisfied if:

$$k^{-1} >> \sqrt{\frac{I_0}{A_0}} = r_0, \tag{28}$$

so that $\mu >> 1$. This relation illustrates why mid-plane stretching nonlinearity is much stronger than all other nonlinearities considered in Section 2.1, since the nonlinear term in (25) is much larger than the nonlinear terms in (14).

Searching a solution to (25) in the form (16) with only the first harmonic taken into account, we obtain the following equation for the new variable $\varphi(x)$:

$$\left[(1+\chi_I \sin x)\varphi''\right]'' - \mu \frac{1}{\int_{\phi}^{l+\phi} \frac{1}{1+\chi_A \sin x} dx} \left(\eta l \varphi'' + \varphi'' \int_{\phi}^{l+\phi} \varphi' \bar{\varphi}' dx + \frac{1}{2} \bar{\varphi}'' \int_{\phi}^{l+\phi} (\varphi')^2 dx \right) - \omega^2 (1+\chi_A \sin x)\varphi = 0 \quad (29)$$

By contrast to Case A the beam length l and phase ϕ are here present in the governing equation for $\varphi(x)$, so that the effect of nonlinearity may depend on these parameters. However, it is expected that for relatively large l, allowing attenuation of waves from band-gaps ranges before reaching the boundaries, this dependency should vanish.

3. Solution by the method of varying amplitudes

3.1. Case A: Beam unrestricted longitudinally; large deflections possible

Conventional methods for analyzing spatially periodic structures, e.g. the classical Hill's method of infinite determinants [10,11] and the method of space-harmonics [12], are not applicable for the considered problem, since they are based on Floquet theory which is valid for linear systems only. Consequently a novel approach, the *Method of Varying Amplitudes* (MVA) [15,16], is employed. In [16] the corresponding linear problem was studied by this method; the problem did not allow an exact solution, since, as in the present case, modulation of the beam structural properties was not required to be small or piecewise constant. Following the method a solution of (17) is sought in the form of series of spatial harmonics with varying amplitudes:

$$\varphi(x) = b_0(x) + b_{11}(x)\exp(ix) + b_{12}(x)\exp(-ix) + b_{21}(x)\exp(2ix) + b_{22}(x)\exp(-2ix) + \ldots, \quad (30)$$

where the complex-valued amplitudes $b_0(x)$, $b_{j1}(x)$, $b_{j2}(x)$, $j = 1, 2, \ldots m$, are not required to vary slowly in comparison with $\exp(ix)$, $\exp(2ix)$ etc. Note that the solution ansatz implied in the MVA, i.e. the choice of harmonics in (30), depends on the parameters of modulation in the equation considered. For the present problem the modulation involves $\sin x = \frac{1}{2i}\left(\exp(ix) - \exp(-ix)\right)$, so that ansatz (30) is employed as will be discussed further in Section 5.1.

The shift from the original dependent variable $\varphi(x)$ to $2m+1$ new variables $b_0(x)$, $b_{j1}(x)$, $b_{j2}(x)$ implies that $2m+1$ equations for these variables are needed. This can be accomplished by introducing constraints in the form of $2m$ additional equations. With the MVA the constraints are introduced in the following way: Substitute (30) into (17) and require $2m$ groups of terms to vanish identically. The last $(2m+1)$'th equation will include all the remaining terms of the original equation. The $2m$ groups of terms are proposed to be the coefficients of the lowest spatial harmonics involved, including the zeroth one, i.e. $\exp(ix)$, $\exp(-ix)$, $\exp(2ix)$ etc.

By contrast to the method of harmonic balance [11,28], truncation of the series (30) does not imply any approximations in itself. For example with only a single term taken into account,

$\varphi(x) = b_0(x)$, the resulting equation for $b_0(x)$ will be the same as the initial equation (17) for $\varphi(x)$, so that nothing is gained by the shift of variables. The usefulness of (30) lies in transforming (17) into a larger number of equations that are more convenient for subsequent solving.

Taking into account only the five written terms in (30) one obtains five equations for the amplitudes $b_0(x)$, $b_{11}(x)$, $b_{12}(x)$, $b_{21}(x)$, $b_{22}(x)$:

$$b_0'''' - \omega^2 b_0 + \frac{\omega^2 \chi_A}{2i}(b_{11} - b_{12}) + \frac{\chi_I}{2i}(b_{11}'' - b_{12}'' - 2ib_{11}''' - 2ib_{12}''' - b_{11}'''' + b_{12}'''') = N_1(\mathbf{b}), \tag{31}$$

$$\begin{aligned} &b_{11}'''' + 4ib_{11}''' - 6b_{11}'' - 4ib_{11}' + (1-\omega^2)b_{11} + \frac{i\omega^2\chi_A}{2}(b_0 - b_{21}) \\ &\quad + \frac{i\chi_I}{2}\left(4b_{21} - 12ib_{21}' + b_0'' - 13b_{21}'' - 2ib_0''' + 6ib_{21}''' - b_0'''' + b_{21}''''\right) = N_2(\mathbf{b}), \end{aligned} \tag{32}$$

$$\begin{aligned} &b_{12}'''' - 4ib_{12}''' - 6b_{12}'' + 4ib_{12}' + (1-\omega^2)b_{12} - \frac{i\omega^2\chi_A}{2}(b_0 - b_{22}) \\ &\quad - \frac{i\chi_I}{2}\left(4b_{22} + 12ib_{22}' + b_0'' - 13b_{22}'' + 2ib_0''' - 6ib_{22}''' - b_0'''' + b_{22}''''\right) = N_3(\mathbf{b}), \end{aligned} \tag{33}$$

$$b_{21}'''' + 8ib_{21}''' - 24b_{21}'' - 32ib_{21}' + (16-\omega^2)b_{21} + \frac{i\omega^2\chi_A}{2}b_{11} - \frac{i\chi_I}{2}\left(4b_{11} - 12ib_{11}' - 13b_{11}'' + 6ib_{11}''' + b_{11}''''\right) = N_4(\mathbf{b}), \tag{34}$$

$$\begin{aligned} &\Bigg[b_{22}'''' - 8ib_{22}''' - 24b_{22}'' + 32ib_{22}' + (16-\omega^2)b_{22} - \frac{i\omega^2\chi_A}{2}b_{12} + \frac{i\chi_I}{2}\left(4b_{12} + 12ib_{12}' - 13b_{12}'' - 6ib_{12}''' + b_{12}''''\right) \\ &\quad - N_5(\mathbf{b})\Bigg]\exp(-2ix) = N_6(\mathbf{b}) - i\left[\chi_I\left(\frac{1}{2}b_{21}'''' + 5ib_{21}''' - \frac{37}{2}b_{21}'' - 30ib_{21}'\right)\right. \\ &\quad \left. - \left(\frac{\omega^2\chi_A}{2} - 18\chi_I\right)b_{21}\right]\exp(3ix) + i\left[\chi_I\left(\frac{1}{2}b_{22}'''' - 5ib_{22}''' - \frac{37}{2}b_{22}'' + 30ib_{22}'\right)\right. \\ &\quad \left. - \left(\frac{\omega^2\chi_A}{2} - 18\chi_I\right)b_{22}\right]\exp(-3ix), \end{aligned} \tag{35}$$

where $\mathbf{b} = \{b_0 \quad b_{11} \quad b_{12} \quad b_{21} \quad b_{22}\}^{\mathbf{T}}$, and $N_j(\mathbf{b})$, $j = 1,\ldots,6$, are nonlinear functions which are rather lengthy, and thus not given here. The term $N_6(\mathbf{b})$ involves harmonics of order three and higher, i.e. $\exp(3ix)$, $\exp(4ix)$ etc. When composing equations (31)-(35), the relation

$$\int g(x)\exp(ijx)dx = G(x)\exp(ijx), \quad j = -5,-4,\ldots,4,5, \tag{36}$$

has been employed, where G and g are related by:

$$\frac{dG(x)}{dx} + ijG(x) - g(x) = 0. \tag{37}$$

Thus we have restated the original equation (17) in the form of five equations (31)-(35), with b_0, b_{11}, b_{12}, b_{21}, b_{22} as the dependent variables instead of φ; no approximations are involved so far. These are introduced in the following step, where we neglect the right-hand side of (35), i.e. harmonic terms of higher order than involved in the substituted series (30), so that:

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$$b_{22}'''' - 8ib_{22}''' - 24b_{22}'' + 32ib_{22}' + (16-\omega^2)b_{22} - \frac{i\omega^2\chi_A}{2}b_{12} + \frac{i\chi_I}{2}\left(4b_{12} + 12ib_{12}' - 13b_{12}'' - 6ib_{12}''' + b_{12}''''\right) = N_5(\mathbf{b}) \quad (38)$$

This simplification is adequate when the right-hand side of (35) is small in comparison with the leading terms of (34)-(35), which is the case if

$$\left|18\chi_I - \frac{\omega^2\chi_A}{2}\right| << \left|\omega^2 - 16\right|, \quad (39)$$

and the involved nonlinearities are weak:

$$\left|N_6(\mathbf{b})\right| << \left|\omega^2 - 16\right|\sqrt{b_{21}^2 + b_{22}^2}\,. \quad (40)$$

Since $\omega = O(1)$ and $0 \le \chi_A < 1$, an additional restriction $\chi_I << 1$ should be imposed to satisfy (39), (in fact it is sufficient to require $\chi_I \le 0.5$, cf. 5.3). So only comparatively small modulations of the beam stiffness can be considered by the means of the method; modulations of the beam mass per unit length, however, can be large.

Equations (31)-(34), (38) are nonlinear differential equations in $\mathbf{b}(x)$. They allow a multitude of solutions, in particular those that can be written in the form

$$\mathbf{b}(x) = \mathbf{b}_c \exp(-i\kappa x)\,. \quad (41)$$

Indeed, substituting (41) into (31)-(34), (38), we obtain that all the nonlinear terms $N_j(\mathbf{b})$, $j = 1,\ldots,5$, are proportional to $\exp(-i\kappa x)$. Consequently, by multiplying these equations by $\exp(i\kappa x)$ they can be reduced to a system of nonlinear algebraic equations in κ and components $b_{11c}, b_{12c}, b_{21c}$, b_{22c} of the constant vector $\mathbf{b}_c$.

In the linear case, i.e. with $N_{1-5} = 0$ in (31)-(34), (38), these equations allow solutions *only* of the form (41), with $-i\kappa$ being a root of the characteristic equation corresponding to (31)-(34), (38), and $\mathbf{b}_c$ the associated vector. Consequently, taking into account (16), (30) and (41), the solution of the linear counterpart of the initial dimensionless equation (14) may be written as:

$$\begin{aligned} &w(x,t) = F(x)\exp\left(i(\omega t - \kappa x)\right) + cc, \\ &F(x) = b_{0c} + b_{11c}\exp(ix) + b_{12c}\exp(-ix) + b_{21c}\exp(2ix) + b_{22c}\exp(-2ix), \end{aligned} \quad (42)$$

where cc denote complex conjugate terms. This solution obeys Floquet theory [1], since $F(x)$ has the same period as the cross-section modulation. It describes a “compound wave” [1] or a “wave package” [12] propagating (or attenuating) in the beam with dimensionless frequency ω and wavenumber κ, with the relation between ω and κ defining the dispersion relation of the considered periodic structure, and with real values of κ corresponding to propagating waves and complex values to attenuating waves [1,12].

Our aim is to examine the effect of nonlinearities on the beam dispersion relation and frequency band-gaps. This implies that we are interested in solutions to (31)-(34), (38) *only* of the form (41), so that the corresponding solution of the initial dimensionless equation (14) takes the form (42) describing propagating (or attenuating) wave with dimensionless frequency ω and wavenumber κ.

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Since $b_{0c}, b_{11c}, b_{12c}, b_{21c}$, and b_{22c} are complex-valued we have to consider also the complex conjugates of (31)-(34), (38). Introducing

$$b_{0c} = \hat{b}_0 e^{i\theta_0},\ b_{11c} = \hat{b}_{11} e^{i\theta_{11}},\ b_{12c} = \hat{b}_{12} e^{i\theta_{12}},\ b_{21c} = \hat{b}_{21} e^{i\theta_{21}},\ b_{22c} = \hat{b}_{22} e^{i\theta_{22}}, \tag{43}$$

where $\hat{b}_0, \hat{b}_{11}, \hat{b}_{12}, \hat{b}_{21}, \hat{b}_{22}$ and $\theta_0, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}$ are real-valued constants, and substituting into (31)-(34), (38) and their complex conjugates, gives:

$$\theta_0 = \theta_{21} = \theta_{22} = \theta, \qquad \theta_{11} = \theta_{12} = \theta - \pi/2, \tag{44}$$

where θ can take arbitrary values without affecting the resulting equations for $\hat{b}_0, \hat{b}_{11}, \hat{b}_{12}, \hat{b}_{21}$, and $\hat{b}_{22}$:

$$\hat{b}_0(\kappa^4 - \omega^2) + \frac{\omega^2 \chi_A}{2}(\hat{b}_{12} - \hat{b}_{11}) + \frac{\kappa^2 \chi_I}{2}\left(\hat{b}_{11}(\kappa-1)^2 - \hat{b}_{12}(\kappa+1)^2\right) = \hat{N}_1, \tag{45}$$

$$\hat{b}_{11}\left((\kappa-1)^4 - \omega^2\right) + \frac{\omega^2 \chi_A}{2}(\hat{b}_{21} - \hat{b}_0) - \frac{(\kappa-1)^2 \chi_I}{2}\left(\hat{b}_{21}(\kappa-2)^2 - \hat{b}_0 \kappa^2\right) = \hat{N}_2, \tag{46}$$

$$\hat{b}_{12}\left((\kappa+1)^4 - \omega^2\right) - \frac{\omega^2 \chi_A}{2}(\hat{b}_{22} - \hat{b}_0) + \frac{(\kappa+1)^2 \chi_I}{2}\left(\hat{b}_{22}(\kappa+2)^2 - \hat{b}_0 \kappa^2\right) = \hat{N}_3, \tag{47}$$

$$\hat{b}_{21}\left((\kappa-2)^4 - \omega^2\right) + \frac{\omega^2 \chi_A}{2}\hat{b}_{11} - \frac{\chi_I}{2}(2 - 3\kappa + \kappa^2)^2 \hat{b}_{11} = \hat{N}_4, \tag{48}$$

$$\hat{b}_{22}\left((\kappa+2)^4 - \omega^2\right) - \frac{\omega^2 \chi_A}{2}\hat{b}_{12} + \frac{\chi_I}{2}(2 + 3\kappa + \kappa^2)^2 \hat{b}_{12} = \hat{N}_5, \tag{49}$$

where $\hat{N}_{1-5}$ are nonlinear in $\hat{b}_0, \hat{b}_{11}, \hat{b}_{12}, \hat{b}_{21}$, and $\hat{b}_{22}$, and depend on κ, ω, χ_I, χ_A, and β_n.

The effect of nonlinearities on the dispersion relation and frequency band-gaps depends on the magnitude of transverse deflections w as given by expression (42), which with (43)-(44) can be written:

$$\begin{aligned} w(x,t) = {} & 2\left(\hat{b}_0 + (\hat{b}_{11} - \hat{b}_{12})\sin x + (\hat{b}_{22} + \hat{b}_{21})\cos 2x\right)\cos\left(\omega t - \kappa x + \theta\right) \\ & + 2\left((\hat{b}_{11} + \hat{b}_{12})\cos x + (\hat{b}_{22} - \hat{b}_{21})\sin 2x\right)\sin\left(\omega t - \kappa x + \theta\right) \end{aligned} \tag{50}$$

Considering (50), we obtain that the spatially averaged amplitude of the beam transverse deflections w is given by

$$B = 2\sqrt{\hat{b}_0^2 + \hat{b}_{11}^2 + \hat{b}_{12}^2 + \hat{b}_{21}^2 + \hat{b}_{22}^2}\,. \tag{51}$$

Consequently the amplitudes $\hat{b}_0, \hat{b}_{11}, \hat{b}_{12}, \hat{b}_{21}$, and $\hat{b}_{22}$ can be compared in magnitude to this value. The algebraic equations (45)-(49) and (51) are then solved for $\hat{b}_0, \hat{b}_{11}, \hat{b}_{12}, \hat{b}_{21}, \hat{b}_{22}$, and κ as functions of the amplitude B and parameters ω, χ_I, χ_A, and β_n. Thus the dispersion relation $\kappa = \kappa(\omega)$ of the considered nonlinear periodic beam is obtained for various values of B, χ_I, χ_A, and β_n, as will be illustrated in Section 4, and discussed and validated in Section 5.

3.2. Case B: Mid-plane stretching

Employing again the MVA and searching a solution of (29) in the form (30) with only the written terms taken into account, one obtains the following equations for the new variables b_0, b_{11}, b_{12}, b_{21}, and b_{22}:

$$b_0'''' - \omega^2 b_0 + \frac{\omega^2 \chi_A}{2i}(b_{11} - b_{12}) + \frac{\chi_I}{2i}(b_{11}'' - b_{12}'' - 2ib_{11}''' - 2ib_{12}''' - b_{11}'''' + b_{12}'''') = (S_1 + S_2)b_0'' + S_3\bar{b}_0'', \tag{52}$$

$$\begin{aligned} & b_{11}'''' + 4ib_{11}''' - 6b_{11}'' - 4ib_{11}' + (1-\omega^2)b_{11} + \frac{i\omega^2\chi_A}{2}(b_0 - b_{21}) + \frac{i\chi_I}{2}\big(4b_{21} - 12ib_{21}' + b_0'' - 13b_{21}'' \\ & \qquad - 2ib_0''' + 6ib_{21}''' - b_0'''' + b_{21}''''\big) = (S_1 + S_2)(b_{11}'' + 2ib_{11}' - b_{11}) + S_3(\bar{b}_{12}'' + 2i\bar{b}_{12}' - \bar{b}_{12}), \end{aligned} \tag{53}$$

$$\begin{aligned} & b_{12}'''' - 4ib_{12}''' - 6b_{12}'' + 4ib_{12}' + (1-\omega^2)b_{12} - \frac{i\omega^2\chi_A}{2}(b_0 - b_{22}) - \frac{i\chi_I}{2}\big(4b_{22} + 12ib_{22}' + b_0'' - 13b_{22}'' \\ & \qquad + 2ib_0''' - 6ib_{22}''' - b_0'''' + b_{22}''''\big) = (S_1 + S_2)(b_{12}'' - 2ib_{12}' - b_{12}) + S_3(\bar{b}_{11}'' - 2i\bar{b}_{11}' - \bar{b}_{11}), \end{aligned} \tag{54}$$

$$\begin{aligned} & b_{21}'''' + 8ib_{21}''' - 24b_{21}'' - 32ib_{21}' + (16-\omega^2)b_{21} + \frac{i\omega^2\chi_A}{2}b_{11} - \frac{i\chi_I}{2}\big(4b_{11} - 12ib_{11}' - 13b_{11}'' + 6ib_{11}''' + b_{11}''''\big) \\ & \qquad = (S_1 + S_2)(b_{21}'' + 4ib_{21}' - 4b_{21}) + S_3(\bar{b}_{22}'' + 4i\bar{b}_{22}' - 4\bar{b}_{22}), \end{aligned} \tag{55}$$

$$\begin{aligned} & \Big[b_{22}'''' - 8ib_{22}''' - 24b_{22}'' + 32ib_{22}' + (16-\omega^2)b_{22} - \frac{i\omega^2\chi_A}{2}b_{12} + \frac{i\chi_I}{2}\big(4b_{12} + 12ib_{12}' - 13b_{12}'' - 6ib_{12}''' + b_{12}''''\big) \\ & \qquad - (S_1 + S_2)(b_{22}'' - 4ib_{22}' - 4b_{22}) - S_3(\bar{b}_{21}'' - 4i\bar{b}_{21}' - 4\bar{b}_{21}) \Big] \exp(-2ix) \\ & \qquad = i\left[\left(\frac{\omega^2\chi_A}{2} - 18\chi_I\right)b_{21} - \chi_I\left(\frac{1}{2}b_{21}'''' + 5ib_{21}''' - \frac{37}{2}b_{21}'' - 30ib_{21}'\right)\right]\exp(3ix) \\ & \qquad - i\left[\left(\frac{\omega^2\chi_A}{2} - 18\chi_I\right)b_{22} - \chi_I\left(\frac{1}{2}b_{22}'''' - 5ib_{22}''' - \frac{37}{2}b_{22}'' + 30ib_{22}'\right)\right]\exp(-3ix), \end{aligned} \tag{56}$$

where

$$S_1 = \frac{\mu}{H}\eta l, \quad S_2 = \frac{\mu}{H}\int_{\phi}^{l+\phi} \varphi'\bar{\varphi}' dx, \quad S_3 = \frac{1}{2}\frac{\mu}{H}\int_{\phi}^{l+\phi} (\varphi')^2 dx, \quad H = \int_{\phi}^{l+\phi} \frac{1}{1 + \chi_A \sin x} dx \tag{57}$$

As in Section 3.1, requiring the modulation of the beam stiffness to be small ($\chi_I << 1$), (56) is reduced to:

$$\begin{aligned} & b_{22}'''' - 8ib_{22}''' - 24b_{22}'' + 32ib_{22}' + (16-\omega^2)b_{22} - \frac{i\omega^2\chi_A}{2}b_{12} + \frac{i\chi_I}{2}\big(4b_{12} + 12ib_{12}' - 13b_{12}'' - 6ib_{12}''' + b_{12}''''\big) \\ & \qquad = (S_1 + S_2)(b_{22}'' - 4ib_{22}' - 4b_{22}) + S_3(\bar{b}_{21}'' - 4i\bar{b}_{21}' - 4\bar{b}_{21}) \end{aligned} \tag{58}$$

A solution of the form (41) cannot satisfy (52)-(55) and (58), due to the presence of the terms multiplied by S_3. Indeed, substituting (41) into these terms one obtains expressions of the form $H(\mathbf{b}_c)\exp(i\kappa x)$, which describe a wave with the same wavenumber κ and frequency ω as the primary one, but propagating in the *opposite* direction. Thus the analysis of the dispersion

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relation of the considered nonlinear beam makes sense only with the requirement for this additional wave to be negligibly weak and not affecting the primary wave. This leads to the condition

$$S_3 << 1, \tag{59}$$

which implies the terms multiplied by S_3 in (52)-(55), (58) to be much smaller than the leading terms, so that the solution form (41) can be employed.

Considering S_1 and S_2 in (57), it is found that for a relatively long beam $l >> 1$, and with the solution form (41), they can be approximated as:

$$S_1 = \mu\eta \frac{1-\chi_A^2+\chi_A^4/8}{1-\chi_A^2/2}, \tag{60}$$

$$\begin{aligned} S_2 = \mu\Big(&\kappa^2 b_{0c}\bar{b}_{0c} + (\kappa-1)^2 b_{11c}\bar{b}_{11c} + (\kappa+1)^2 b_{12c}\bar{b}_{12c} \\ &+ (\kappa-2)^2 b_{21c}\bar{b}_{21c} + (\kappa+2)^2 b_{22c}\bar{b}_{22c}\Big)\frac{1-\chi_A^2+\chi_A^4/8}{1-\chi_A^2/2}. \end{aligned} \tag{61}$$

Requiring then (59) to be satisfied, and introducing real constants $\hat{b}_0, \hat{b}_{11}, \hat{b}_{12}, \hat{b}_{21}, \hat{b}_{22}$ and $\theta_0, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}$ according to (43), one obtains relations (44) as for Case A, and the following equations for $\hat{b}_0, \hat{b}_{11}, \hat{b}_{12}, \hat{b}_{21}, \hat{b}_{22}$:

$$\hat{b}_0(\kappa^4-\omega^2) + \frac{\omega^2\chi_A}{2}(\hat{b}_{12}-\hat{b}_{11}) + \frac{\kappa^2\chi_I}{2}\left(\hat{b}_{11}(\kappa-1)^2 - \hat{b}_{12}(\kappa+1)^2\right) = -\kappa^2\hat{b}_0(S_1+S_2), \tag{62}$$

$$\hat{b}_{11}\left((\kappa-1)^4-\omega^2\right) + \frac{\omega^2\chi_A}{2}(\hat{b}_{21}-\hat{b}_0) - \frac{(\kappa-1)^2\chi_I}{2}\left(\hat{b}_{21}(\kappa-2)^2 - \hat{b}_0\kappa^2\right) = -(\kappa-1)^2\hat{b}_{11}(S_1+S_2), \tag{63}$$

$$\hat{b}_{12}\left((\kappa+1)^4-\omega^2\right) - \frac{\omega^2\chi_A}{2}(\hat{b}_{22}-\hat{b}_0) + \frac{(\kappa+1)^2\chi_I}{2}\left(\hat{b}_{22}(\kappa+2)^2 - \hat{b}_0\kappa^2\right) = -(\kappa+1)^2\hat{b}_{12}(S_1+S_2), \tag{64}$$

$$\hat{b}_{21}\left((\kappa-2)^4-\omega^2\right) + \frac{\omega^2\chi_A}{2}\hat{b}_{11} - \frac{\chi_I}{2}(2-3\kappa+\kappa^2)^2\hat{b}_{11} = -(\kappa-2)^2\hat{b}_{21}(S_1+S_2), \tag{65}$$

$$\hat{b}_{22}\left((\kappa+2)^4-\omega^2\right) - \frac{\omega^2\chi_A}{2}\hat{b}_{12} + \frac{\chi_I}{2}(2+3\kappa+\kappa^2)^2\hat{b}_{12} = -(\kappa+2)^2\hat{b}_{22}(S_1+S_2), \tag{66}$$

where for S_1 the approximation (60) can be used; and (61) for S_2 takes the form:

$$S_2 = \mu\left(\kappa^2\hat{b}_0^2 + (\kappa-1)^2\hat{b}_{11}^2 + (\kappa+1)^2\hat{b}_{12}^2 + (\kappa-2)^2\hat{b}_{21}^2 + (\kappa+2)^2\hat{b}_{22}^2\right)\frac{1-\chi_A^2+\chi_A^4/8}{1-\chi_A^2/2}. \tag{67}$$

As is seen (62)-(66) do not involve the beam length l and phase ϕ so that, as was predicted (cf. Section 2.2), the dispersion relation of the considered nonlinear beam does not depend on these parameters. The effect of the nonlinearity on this relation is governed by the term S_2, present in (62)-(66). Comparing expression (67) for S_2 with (60) for S_1 it appears they differ only at the position of the initial pre-stretching coefficient η; hence the nonlinearity is equivalent to an additional stretching of the beam:

$$\eta_n = \kappa^2\hat{b}_0^2 + (\kappa-1)^2\hat{b}_{11}^2 + (\kappa+1)^2\hat{b}_{12}^2 + (\kappa-2)^2\hat{b}_{21}^2 + (\kappa+2)^2\hat{b}_{22}^2, \tag{68}$$

which depends on the magnitude of the beam transverse deflections w; for propagating waves (real values of κ) $\eta_n > 0$.

Thus the approximate solution of the initial dimensionless equation (29) is obtained in the form (50), describing a propagating (or attenuating) wave with dimensionless frequency ω and wavenumber κ. As for Case A the amplitude B is introduced by (51) to define the magnitudes of $\widehat{b}_0, \widehat{b}_{11}, \widehat{b}_{12}, \widehat{b}_{21}$, and $\widehat{b}_{22}$. Equations (62)-(66) and (51) are then solved for $\widehat{b}_0, \widehat{b}_{11}, \widehat{b}_{12}, \widehat{b}_{21}, \widehat{b}_{22}$, and κ as functions of the amplitude B and parameters ω, χ_I, χ_A, μ, and η.

4. Dispersion relations and frequency band-gaps

4.1. Effects of nonlinear (true) curvature and nonlinear material

To simplify the analysis of the effects of nonlinearities on the dispersion relation, we first consider each source of nonlinearity separately, in this section nonlinear (true) curvature, and nonlinear material. Effects of nonlinear inertia are not taken into account, but will be studied in Section 4.3.

To illustrate the effect of nonlinear (true) curvature, Figure 1(a-c) show the dispersion relation $\kappa(\omega)$ for a linear material ($\beta_n = 0$) and various values of the amplitude B and modulation amplitudes χ_I, χ_A, with the linear dispersion relation shown for comparison (dotted line). According to [16], for pure modulation χ_A of the beam mass per unit length the linear dispersion relation features two distinct band-gaps in the considered frequency range (at $\omega \approx 0.25$ and $\omega \approx 1$), while pure modulation χ_I of the beam stiffness gives one bandgap (at $\omega \approx 0.25$), and modulations with equal amplitudes $\chi_A = \chi_I$ also one bandgap (at $\omega \approx 1$). These results can be seen from Figure 1(a-c) (dotted line) with the nonlinear dispersion relation (solid line) slightly shifted to higher frequencies.

Figure 1(d, e) present the dispersion relation with only the effect of nonlinear material being taken into account. Note that the parameter β_n, governing the nonlinearity of the beam's material stress-strain relation, is not necessarily small, since $\beta_n = \tilde{\beta}_n k^2$ and the spatial frequency k of the modulation can be large. As appears the nonlinear dispersion relation (solid line) is shifted to lower frequencies, so that the effect of this nonlinearity is opposite to the one of nonlinear curvature. Figure 1(f-h) represent the case when both sources of nonlinearity are taken into account. As is seen solid and dashed lines almost coincide, so that it is possible to compensate the effect of nonlinear curvature by nonlinear material.

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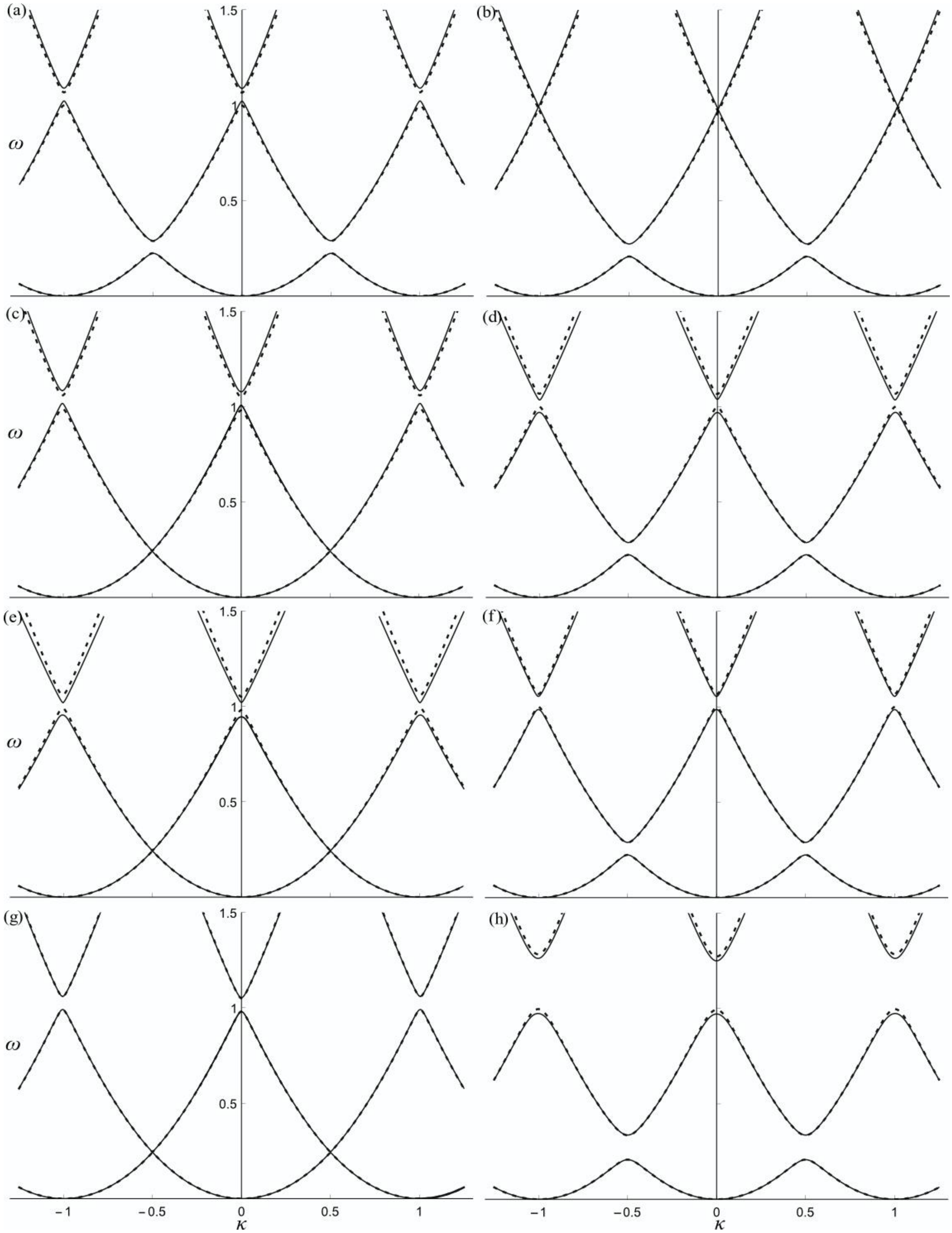

Figure 1. Dispersion relations $\omega(\kappa)$ for (a-c) $\beta_n = 0$ and (a) $B = 0.5$, $\chi_A = 0.5$, $\chi_I = 0$; (b) $B = 0.45$, $\chi_A = 0$, $\chi_I = 0.5$; (c) $B = 0.55$, $\chi_A = \chi_I = 0.5$; (d, e) only the effect of nonlinear material taken into account, $\beta_n = 0.25$, and (d) $B = 0.45$,

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$\chi_A = 0.5$, $\chi_I = 0$; (e) $B = 0.5$, $\chi_A = \chi_I = 0.5$; (f, g) both sources of nonlinearity taken into account (f) $\beta_n = 0.25$, $B = 0.5$, $\chi_A = 0.5$, $\chi_I = 0$; (g) $\beta_n = 0.15$, $B = 0.4$, $\chi_A = \chi_I = 0.5$; (h) $\beta_n = 0.35$, $B = 0.45$, $\chi_A = 0.9$, $\chi_I = 0$. Solid lines: nonlinear beam; dotted: linear beam.

As appears from Figure 1 the structure of the beam dispersion relation does not change due to the nonlinearities, i.e. the relation remains periodic with respect to κ, and the number of band-gaps is the same as in the linear case. On the other hand, a shift of the band-gaps to a higher (or lower) frequency range is revealed. And even for weak nonlinearities this may cause frequency band-gaps, i.e. frequency ranges in which waves *cannot* propagate, to become pass-bands, i.e. frequency ranges in which waves *can* propagate, and vice versa, cf. Figure 1(c-e). The effects of nonlinearities are more pronounced for higher frequencies and the corresponding band-gaps. The lumped parameter models used in [13,14] to study nonlinear dispersion relations were able to capture only the lowest band-gap, and thus did not reveal this effect.

According to the phase closure principle [25], frequencies corresponding to boundaries of band-gap regions for linear periodic structures are those where an integer number n of compound half-waves fit exactly into a unit cell of the structure, i.e. they have wavenumbers

$$\kappa = \frac{n}{2}, \quad n = \pm 1, \pm 2, \pm 3, \ldots \tag{69}$$

As follows from (45)-(49) this holds also for the considered nonlinear periodic beam (cf. Section 5.2). So the critical frequencies ω_c, determining boundaries of the band-gaps, can be obtained by letting $\kappa = n/2$, $n = \pm 1, \pm 2, \pm 3, \ldots$ in these equations. As an illustration Figure 2(a, b) show the dependencies of ω_c corresponding to the first ($n = 1$) and the second ($n = 2$) band-gap on the amplitude B for $\beta_n = 0$ and $\chi_A = 0.5$, $\chi_I = 0$. Figure 2(c, d) represent these dependencies for the second band-gap with only the effect of nonlinear material taken into account; as appears the effect of this nonlinearity is the same for different values of the modulation amplitudes χ_A and χ_I. Figure 2(e, f) correspond to the case of combined nonlinearities. As appears from Figure 2, the width of band-gaps is relatively insensitive to (weak) nonlinearities.

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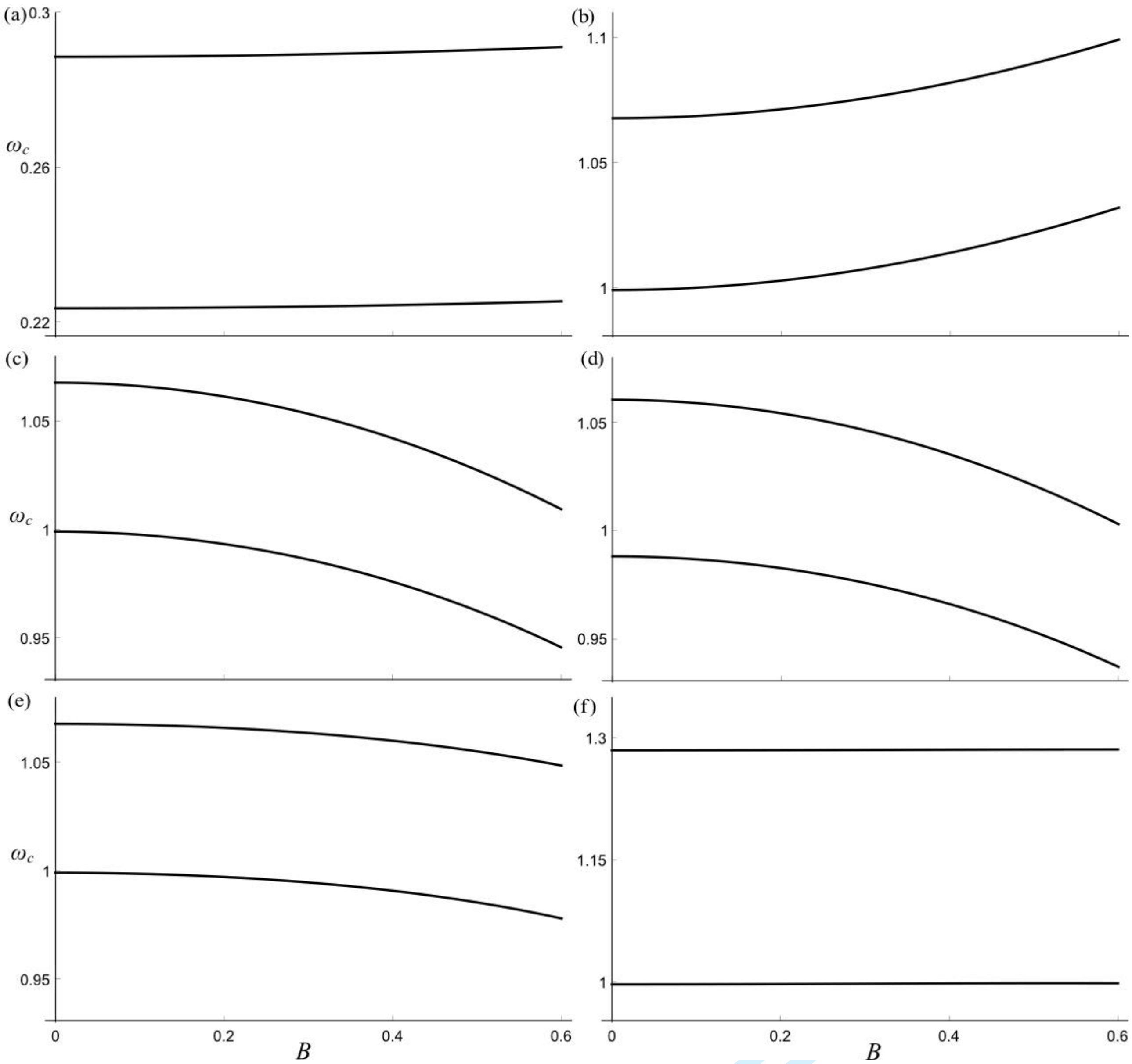

Figure 2. Dependencies of the critical frequencies ω_c, determining band-gap boundaries, on the amplitude B for (a, b) $\beta_n = 0$, $\chi_A = 0.5$, $\chi_I = 0$, (a) first band-gap, $n = 1$; (b) second band-gap, $n = 2$; (c, d) only the effect of nonlinear material taken into account, $\beta_n = 0.25$, $n = 2$, (c) $\chi_A = 0.5$, $\chi_I = 0$; (d) $\chi_A = \chi_I = 0.5$; (e, f) both sources of nonlinearity taken into account, $n = 2$, (e) $\beta_n = 0.25$, $\chi_A = 0.5$, $\chi_I = 0$; (f) $\beta_n = 0.15$, $\chi_A = 0.9$, $\chi_I = 0$.

4.2. Effects of initial pre-stretching and mid-plane stretching nonlinearity

Next consider the case when both ends of the beam are restricted to move longitudinally (Case B). First we analyze the *linear* dispersion relation and the effects of initial pre-stretching η of the beam. Figure 3 illustrates the dispersion relation of the linear beam for $\mu = 100$ and various values of pre-stretching coefficient η and modulation amplitudes χ_I, χ_A; the dispersion relation without pre-stretching is shown for comparison (dotted line).

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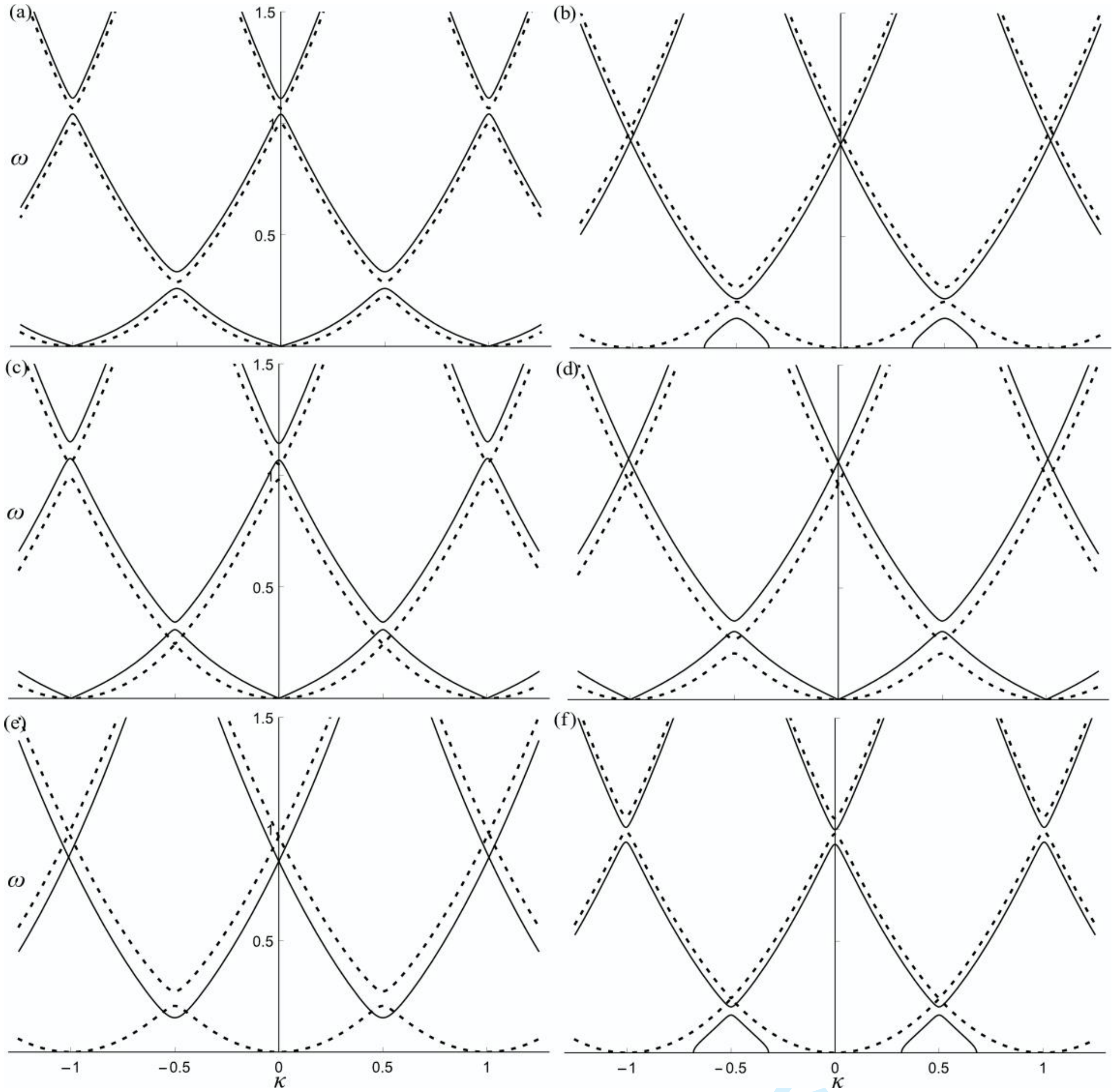

Figure 3. Dispersion relations $\omega(\kappa)$ for the linear pre-stretched (solid line), and unstretched (dotted) beam for $\mu = 100$ and (a) $\eta = 0.001$, $\chi_A = 0.5$, $\chi_I = 0$; (b) $\eta = -0.001$, $\chi_A = 0$, $\chi_I = 0.5$; (c) $\eta = 0.002$, $\chi_A = \chi_I = 0.5$; (d) $\eta = 0.002$, $\chi_A = 0$, $\chi_I = 0.5$; (e) $\eta = -0.002$, $\chi_A = 0$, $\chi_I = 0.5$; (f) $\eta = -0.001$, $\chi_A = \chi_I = 0.5$.

From Figure 3(c, f) it appears that for $\chi_A = \chi_I$, when the unstretched beam does not feature a band-gap at $\omega \approx 0.25$, the pre-stretched beam has. Also, positive pre-stretching appears to shift the band-gaps to higher frequencies (Figure 3(a, d)), and negative pre-stretching to lower frequencies (Figure 3(b)), the effect being most pronounced for low frequencies. In particular it is possible to shift one of the boundaries of the lowest band-gap to zero frequency, as seen in Figure 3 (e); in this case the width of the band-gap is also increased considerably.

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Figure 4 illustrates the dependency of the critical frequencies ω_c, determining boundaries of the band-gaps, on the value of the initial pre-stretching η for $\mu = 100$ and various values of the modulation amplitudes χ_I, χ_A. As appears from Figure 4 (b, d, f) the width of the second band-gap is relatively weakly affected by pre-stretching, though it can be effectively shifted to a higher or lower frequency range. The width of the first band-gap, by contrast, is strongly affected by pre-stretching. For pure modulation of the beam mass per unit length positive pre-stretching increases the band-gap, while negative decreases it as seen in Figure 4(a), and at a certain value of η the width of the band-gap essentially vanishes. In the case of pure modulation of the beam stiffness Figure 4(c), the effect of pre-stretching is opposite, and it is even possible to obtain a large band-gap with zero frequency as the lower boundary. If modulations with equal amplitudes are imposed, Figure 4(e), then negative as well as positive pre-stretching increases the band-gap.

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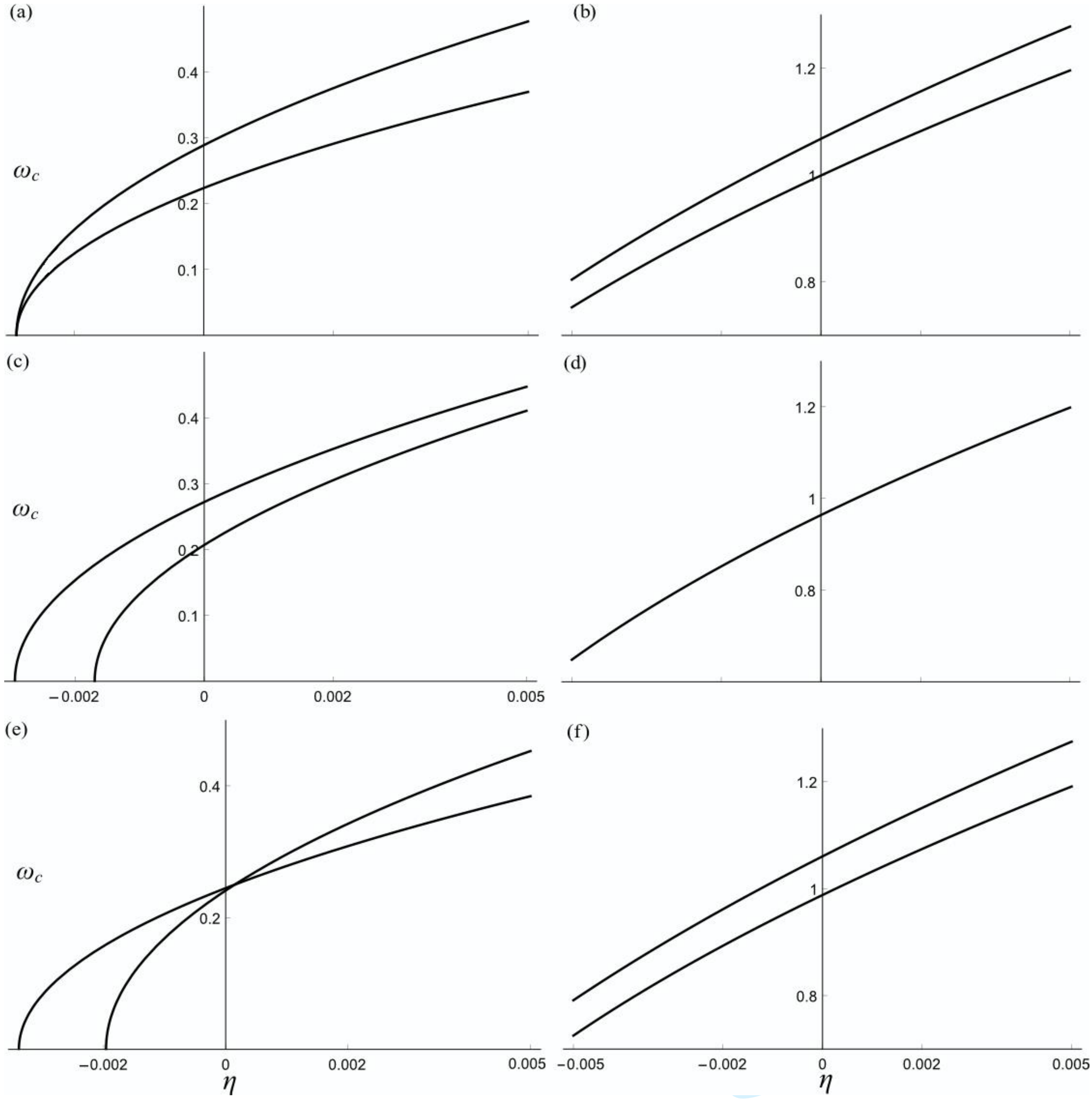

Figure 4. Dependency of the critical frequencies ω_c, determining boundaries of the band-gaps, on the initial pre-stretching η for $\mu = 100$ and (a, b) $\chi_A = 0.5$, $\chi_I = 0$, (a) first band-gap, $n = 1$; (b) second band-gap, $n = 2$; (c, d) $\chi_A = 0$, $\chi_I = 0.5$, (c) first band-gap, $n = 1$; (d) second band-gap, $n = 2$; (e, f) $\chi_A = \chi_I = 0.5$, (e) first band-gap, $n = 1$; (f) second band-gap, $n = 2$.

Considering the isolated effect of mid-plane stretching nonlinearity, it is found that it is similar to the one of nonlinear curvature: the band-gaps are shifted to higher frequencies, while the width of the band-gaps is changed only slightly, as illustrated by Figure 5(a-c). However, this source of nonlinearity is much stronger than nonlinear curvature, being pronounced already at very small values $B \sim 10^{-2}$ of transverse beam deflections.

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As predicted in Section 3.2, a wave with the same wavenumber κ and frequency ω as the primary one, but propagating in the opposite direction, emerges due to mid-plane stretching. This wave appears to be most pronounced for wavenumbers $\kappa = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$, when the linear dispersion relation features band-gaps; in this case the term S_3 governing this wave (cf. (57)) attains its maximum value:

$$|S_3| = \tfrac{1}{2}|S_2|, \tag{70}$$

so that for condition (59) to be fulfilled all the nonlinearities involved should be small, $S_2 \ll 1$.

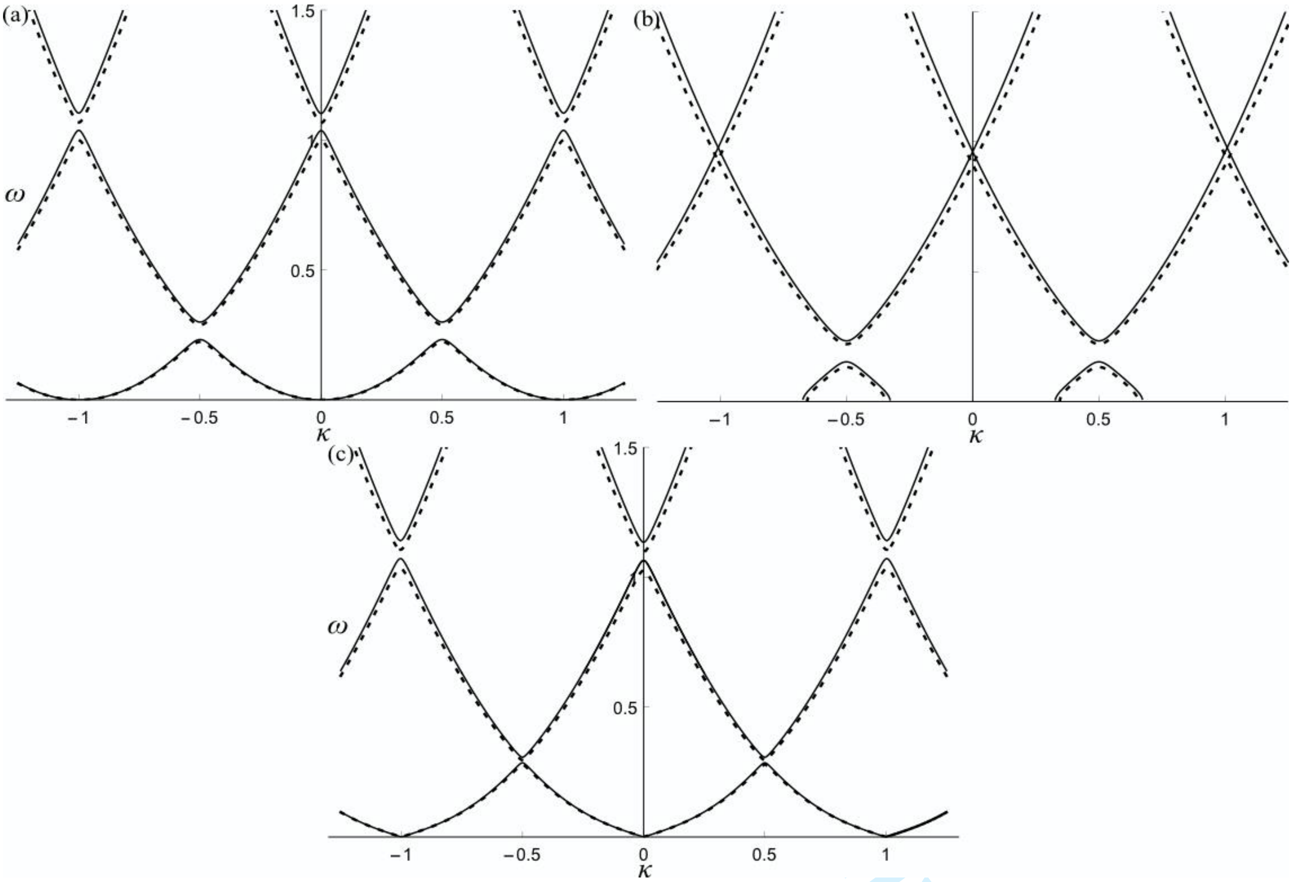

Figure 5. Dispersion relations $\omega(\kappa)$ for the nonlinear (solid line) and linear (dotted) beam for $\mu = 100$, $B = 0.06$ and (a) $\eta = 0$, $\chi_A = 0.5$, $\chi_I = 0$; (b) $\eta = -0.001$, $\chi_A = 0$, $\chi_I = 0.5$; (c) $\eta = 0.001$, $\chi_A = \chi_I = 0.5$.

4.3. Effects of nonlinear inertia

Now consider the effect of nonlinear inertia, governed by the term $(Nw')'$ in (14), on the dispersion relation. This nonlinearity is involved in (14) along with nonlinear curvature and nonlinear material, however, we discuss it separately, since the effects it is causing differ considerably from those already described in Section 4.1.

Substituting the obtained solution w_l for the *linear* beam problem into $(Nw')'$, one finds that for integer values of 2κ and $\chi_A \neq 0$ or $\chi_I \neq 0$ the term tends to infinity for arbitrarily small values of the amplitude B. Thus the dispersion relation for the non-uniform beam should change

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considerably for wavenumbers close to $\kappa = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, ...$ due to nonlinear inertia; it is exactly at these wavenumbers the frequency band-gaps arise in the linear case.

Dispersion relations, relating a certain frequency with a certain wavenumber, as well as frequency band-gaps are features of linear or weakly nonlinear waves only. For strongly nonlinear waves, comprising many components with different frequencies, these notions make no sense [1,12]. For example, it is impossible or rather cumbersome to achieve attenuation of all components of such a wave by periodicity effects. Consequently, in what follows dispersion relations will be presented only for weakly nonlinear waves. A 1:3 ratio between maximum absolute values of the nonlinear term $(Nw')'$ in (14) and the linear term $\left[(1+\chi_I \sin x) w''\right]''$ has been chosen as a threshold to separate weakly nonlinear from strongly nonlinear wave motion. Recall also that the requirement for nonlinearities to be weak has been employed already in Section 3 for solving the initial equations by the MVA.

Figure 6 illustrates dispersion relations of the considered beam with the isolated effect of nonlinear inertia taken into account, for various values of the amplitude B and modulation amplitudes χ_I, χ_A, with the linear dispersion relation shown for comparison in dotted line. Regions in which the wave motion becomes strongly nonlinear are bounded by dashed lines; in these regions nonlinear dispersion relation is not presented. Figure 6(a, b, e-g) correspond to the case of pure modulation of the beam mass per unit length, Figure 6(c) to pure modulation of the beam stiffness, and Figure 6(d, h, i) to modulations with equal amplitudes. As appears the nonlinear dispersion relation does not feature frequency band-gaps in all the considered cases. Instead of the band-gaps, relatively narrow frequency ranges arise in which the wave motion is strongly nonlinear, cf. Figure 6(b, f-h). These frequency ranges correspond to wavenumbers slightly shifted from $\kappa = 0, \pm\frac{1}{2}, \pm 1$, and the size of the shift depends on the amplitude B: the larger the amplitude, the larger the shift; compare e.g. Figure 6(a, b) for $B = 0.4$ with Figure 6(e),(f) for $B = 0.1$. As appears from Figure 6(g, h) the effect of nonlinear inertia is still strong for relatively small beam deflections, $B \sim 10^{-2}$, so that the dispersion relation does not feature frequency band-gaps. For even smaller deflections, $B \sim 10^{-3}$, the size of the shift becomes essentially zero, and strongly nonlinear wave motion is present at $\kappa = 0, \pm\frac{1}{2}, \pm 1$.

A frequency range implying a strongly nonlinear wave motion arises near $\kappa = \pm\frac{1}{2}$ also in the case of modulations with equal amplitudes, $\chi_A = \chi_I$, when there is no band-gap in the linear dispersion relation at $\kappa = \pm\frac{1}{2}$, cf. Figure 6(d, i); this range is for wavenumbers slightly shifted from $\kappa = \pm\frac{1}{2}$, e.g. in Figure 6(d) it is present for $\kappa \approx 0.55$, and the size of the shift depends on the amplitude B in a similar way as described above. Consequently, for $\chi_A = \chi_I$, when the corresponding linear dispersion relation does not feature frequency band-gap for wavenumber $\kappa = \pm\frac{1}{2}$, for very small beam deflections, $B \sim 10^{-3}$, the wave motion at $\kappa = \pm\frac{1}{2}$ becomes strongly nonlinear.

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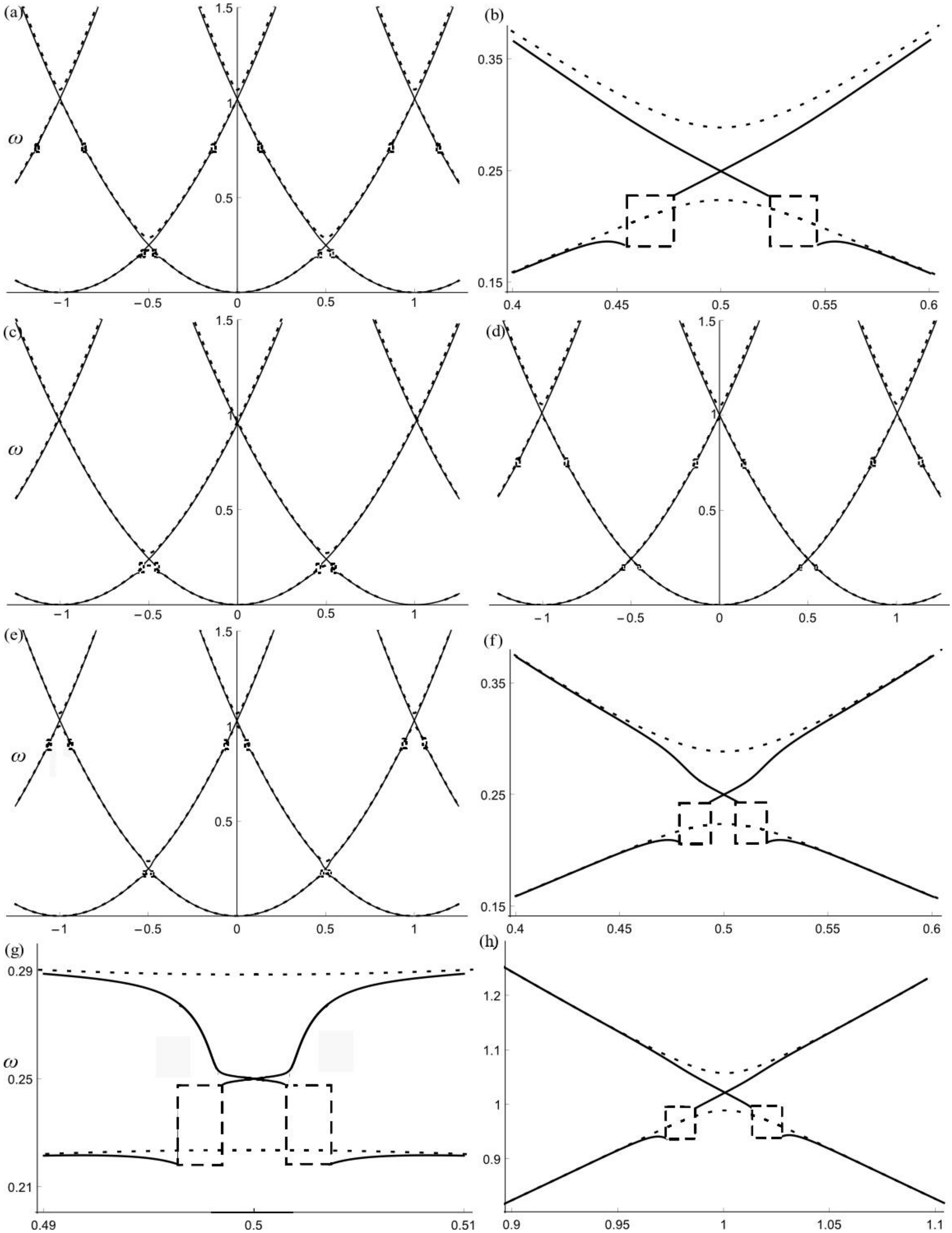

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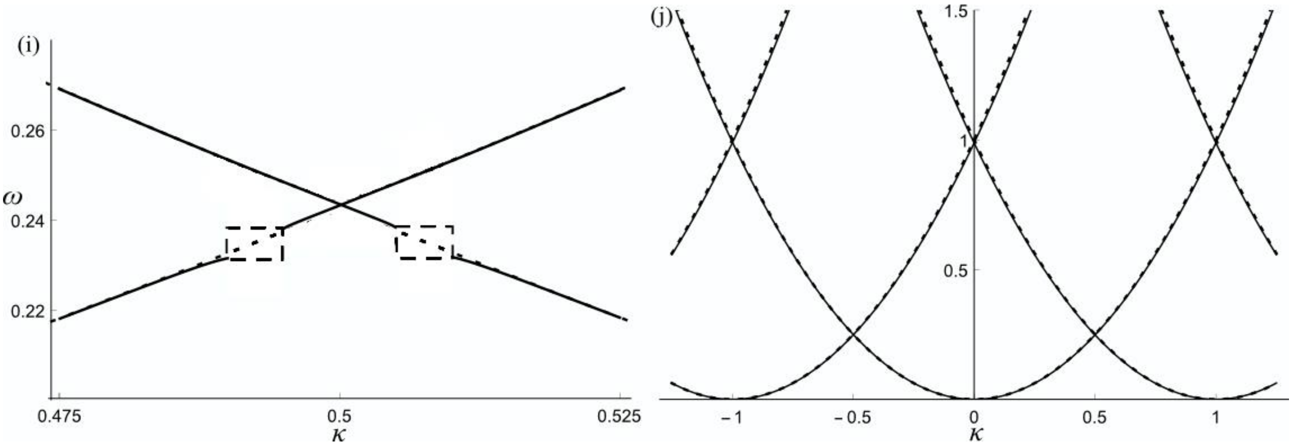

Figure 6. Dispersion relations $\omega(\kappa)$ with only the effect of nonlinear inertia taken into account for (a-d) $B = 0.4$ and (a, b) $\chi_A = 0.5$, $\chi_I = 0$; (c) $\chi_A = 0$, $\chi_I = 0.5$; (d) $\chi_A = \chi_I = 0.5$; (e, f) $B = 0.1$, $\chi_A = 0.5$, $\chi_I = 0$; (g) $B = 0.01$, $\chi_A = 0.5$, $\chi_I = 0$; (h, i) $B = 0.02$, $\chi_A = \chi_I = 0.5$; (j) $B = 0.4$, $\chi_A = \chi_I = 0$. Solid line: nonlinear beam, dotted: linear beam; dashed: regions in which the wave motion is strongly nonlinear.

The effects described above are present for the non-uniform beam only; for the uniform beam the influence of nonlinear inertia on the dispersion relation is much weaker, cf. Figure 6(j). For example, to obtain a strongly nonlinear wave motion, the beam deflections should be much larger than those considered previously, e.g. $B \sim 1$.

The obtained results clearly indicate that nonlinear inertia has a substantial impact on the non-uniform beams dispersion relation. It appears that it removes all the band-gaps, by making the wave motion either strongly nonlinear (for very small beam deflections, $B \sim 10^{-3}$), or weakly nonlinear and featuring no band-gaps (for larger beam deflections). The effects of nonlinear inertia seem to prevail over other nonlinearities considered in Section 4.1, as illustrated by Figure 7, showing the dispersion relation with all three sources of the nonlinearity taken into account.

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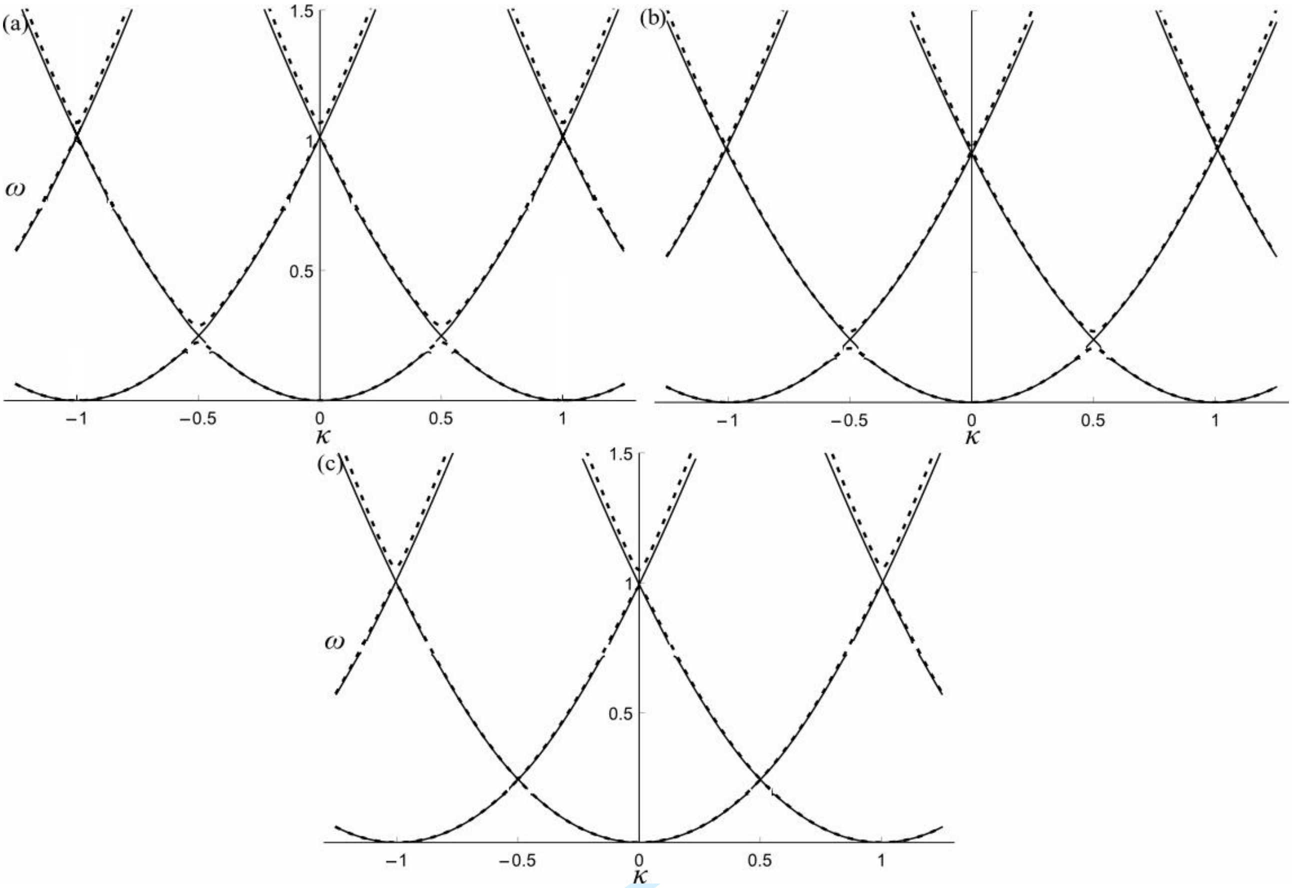

Figure 7. Dispersion relations $\omega(\kappa)$ with the effects of nonlinear inertia, nonlinear curvature and nonlinear material taken into account for $B = 0.4$ and (a) $\beta_n = 0.25$, $\chi_A = 0.5$, $\chi_I = 0$; (b) $\beta_n = 0.2$, $\chi_A = 0$, $\chi_I = 0.5$; (c) $\beta_n = 0.3$, $\chi_A = 0.5$, $\chi_I = 0.5$. Solid line: nonlinear beam, dotted: linear beam.

From the results obtained it follows that real periodic beam structures with *continuous* modulations of parameters performing bending oscillations should not feature frequency band-gaps. In the case of *piecewise constant* modulations, however, the effects of nonlinear inertia can be much weaker, as is suggested by the results obtained for the uniform beam. So such beams *can* feature frequency band-gaps, what was also shown by laboratory experiments, see e.g. [12].

5. Discussion and validation of the results

5.1. On the method of varying amplitudes

As appears from Section 3 application of the MVA for the problem considered requires the presence of the small parameter χ_I in the governing equations, similarly to the asymptotic methods. And the nonlinearities are assumed to be weak which is also typical for these methods. However, since χ_A is not required to be small, the problem considered involves strong parametric excitation. Asymptotic methods, in particular the multiple scales perturbation method [21], are not applicable for such problems. Also, the governing equations (17) and (29) are integro-differential equations, and solving such equations by standard asymptotic methods is very cumbersome [21].

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The MVA involves two solution steps: 1) A shift from the original dependent variable to $2m+1$ new variables $b_0(x)$, $b_{j1}(x)$, $b_{j2}(x)$; 2) approximate solution of the equations for these new variables. The first step does not imply any approximations, however, the equations obtained for the new variables should be suitable for the subsequent approximate solving. The approximation implied in the second step of the method is concerned with neglecting higher order harmonics, similarly to the method of harmonic balance (MHB) [11,28]. Thus, similarly to the MHB, the number of terms in the solution series employed should be sufficient for this approximation, i.e. higher order harmonic terms in the equations for the new variables should be small in comparison with the leading terms. For example, solving equation (29) by the MVA and taking into account only the first three terms in the solution series (30), i.e. not five terms as in the analysis of Section 3, one obtains the following equations for $b_0(x)$, $b_{11}(x)$, $b_{12}(x)$:

$$b_0'''' - \omega^2 b_0 + \frac{\omega^2 \chi_A}{2i}(b_{11} - b_{12}) + \frac{\chi_I}{2i}(b_{11}'' - b_{12}'' - 2ib_{11}''' - 2ib_{12}''' - b_{11}'''' + b_{12}'''') = (S_1 + S_2)b_0'' + S_3\bar{b}_0'', \tag{71}$$

$$\begin{aligned} &b_{11}'''' + 4ib_{11}''' - 6b_{11}'' - 4ib_{11}' + (1-\omega^2)b_{11} + \frac{i\omega^2\chi_A}{2}b_0 + \frac{i\chi_I}{2}\left(b_0'' - 2ib_0''' - b_0''''\right) - (S_1+S_2)(b_{11}'' + 2ib_{11}' - b_{11}) \\ &\quad = S_3(\bar{b}_{12}'' + 2i\bar{b}_{12}' - \bar{b}_{12}), \end{aligned} \tag{72}$$

$$\begin{aligned} &\left[b_{12}'''' - 4ib_{12}''' - 6b_{12}'' + 4ib_{12}' + (1-\omega^2)b_{12} - \frac{i\omega^2\chi_A}{2}b_0 - \frac{i\chi_I}{2}\left(b_0'' + 2ib_0''' - b_0''''\right) - (S_1+S_2)(b_{12}'' - 2ib_{12}' - b_{12}) \right.\\ &\quad \left. - S_3(\bar{b}_{11}'' - 2i\bar{b}_{11}' - \bar{b}_{11})\right]\exp(-ix) = i\left[\frac{\omega^2\chi_A}{2}b_{11} - \frac{\chi_I}{2}\left(4b_{11} - 12ib_{11}' - 13b_{11}'' + 6ib_{11}''' + b_{11}''''\right)\right]\exp(2ix) \\ &\quad - i\left[\frac{\omega^2\chi_A}{2}b_{12} - \frac{\chi_I}{2}\left(4b_{12} + 12ib_{12}' - 13b_{12}'' - 6ib_{12}''' + b_{12}''''\right)\right]\exp(-2ix). \end{aligned} \tag{73}$$

Exact solution of (71)-(73) gives exact solution of the governing equation (29). However, we are interested in the approximate solution, which implies neglecting the higher order harmonic terms in (73). For these terms to be small in comparison with the leading terms of (72)-(73) the following should hold:

$$\left|1-\omega^2\right| >> \left|\frac{\omega^2\chi_A}{2} - 2\chi_I\right|. \tag{74}$$

However, since $\omega = O(1)$ this condition is not fulfilled, and thus more harmonics should be taken into account in the series (30) as was actually done in Section 3. Also, the requirement for the modulation of the beam stiffness to be small ($\chi_I << 1$) is still necessary, even with arbitrarily large number of terms taken into account in (30).

The solution ansatz implied in the MVA, i.e. the harmonics involved in (30), is chosen in such a way to be suitable for the subsequent approximate solution of the equations for the new variables. For example, for (29) the following solution ansatz is also convenient:

$$\varphi(x) = b_{11}(x)\exp(i\tfrac{1}{2}x) + b_{12}(x)\exp(-i\tfrac{1}{2}x) + b_{21}(x)\exp(i\tfrac{3}{2}x) + b_{22}(x)\exp(-i\tfrac{3}{2}x) + \dots . \tag{75}$$

In general, the basic functions in the solution series implied in the MVA can be any orthogonal and complete set of functions, e.g. also Bessel functions.

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As was noted in Section 3 the obtained solution (42) is of the same form as in the linear case, and describes a “compound wave” [1] or a “wave package” [12] propagating (or attenuating) in the beam with dimensionless frequency ω and wavenumber κ. Thus the obtained approximate solution obeys Floquet theory [1], and the relation of the MVA with the classical Hill’s infinite determinant method [1,10,11] and other methods based on this theory becomes apparent. In the general case, i.e. with an infinite number of harmonics taken into account in the series (30), the solution form (42) would become similar to the one implied in the method of space-harmonics [12]. Note, however, that by contrast to these methods the compliance of the approximate solution with Floquet theory was not assumed a priori, but obtained as a result of solving by the MVA, and specific conditions under which Floquet theory holds were determined, e.g. condition (59) for Case B in Section 3.2.

5.2. Some properties of the predicted wave motion

5.2.1. Compliance with the phase closure principle

In Section 4.1 it was noted that the phase closure principle [25] holds for the calculated weakly nonlinear wave motion in the beam. This follows immediately from the fact that the determined approximate solution (42) is of the same form as in the linear case and obeys Floquet theory.

5.2.2. Propagation constant and Bloch parameter

From (42), for the propagation constant p [1,12], describing how a travelling wave changes when passing through a single periodic cell, one obtains:

$$p = \exp\left(2\pi\kappa_B\right) = \exp\left(-i2\pi\kappa\right), \tag{76}$$

where κ_B is the Bloch parameter [1,12], which by (76) becomes:

$$\kappa_B = -i\kappa, \tag{77}$$

so that real values of κ correspond to propagating waves, and complex values to attenuating waves.

5.2.3. Buckling instability

As is well known [11] a negative pre-stretching (i.e. compression) of a beam can result in buckling instability. More specifically, only waves with lengths smaller than a certain critical value can be sustained by such a beam. As appears from Figures 3 and 5 this effect is captured by the solution obtained, since in the case of negative pre-stretching ($\eta < 0$) non-zero values of the wavenumber κ correspond to zero frequency ω.

5.2.4. Possible effects of nonlinear inertia for Case B

From Section 4.3 it follows that the effects of nonlinear inertia on the non-uniform beam dispersion relation can be pronounced even at very small transverse deflections. Thus this nonlinearity is of the same order as mid-plane stretching nonlinearity, and should be taken into account also in Case B when both ends of the beam are restricted to move in the longitudinal direction. Similarly to the case considered in Section 4.3, for a beam with *continuous* modulations of parameters, this can result in vanishing of all the band-gaps due to nonlinear inertia.

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5.3. Validation of results

As appears from Sections 2 and 3 the obtained analytical solution involves two approximations. The first one is concerned with the truncation of the series (16), and the second is implied in the MVA and discussed in Section 5.1. Both of them imply certain nonlinear terms in the equations considered to be discarded. This simplification is valid under the condition that the involved nonlinearities are weak, which is the key assumption of the conducted analysis, carefully checked for all results presented. Cases where the wave motion becomes strongly nonlinear are revealed and discussed separately in Section 4.3. The MVA implies also certain *linear* terms to be discarded, cf. Section 3, equation (38); as is shown this is valid under condition (39), which leads to the requirement for modulations of the beam stiffness to be small, $\chi_I \ll 1$.

To further validate the results a series of numerical experiments was conducted. The initial non-dimensional governing equation (14) (or (25)) was numerically integrated directly using Wolfram Mathematica 7.0 (NDSolve), with periodic boundary conditions and the following initial conditions imposed:

$$\begin{aligned} w(x,0) &= 2\Big[\Big(\hat{b}_0 + (\hat{b}_{11} - \hat{b}_{12})\sin x + (\hat{b}_{22} + \hat{b}_{21})\cos 2x\Big)\cos\kappa x - \Big((\hat{b}_{11} + \hat{b}_{12})\cos x + (\hat{b}_{22} - \hat{b}_{21})\sin 2x\Big)\sin\kappa x\Big], \\ \frac{\partial w}{\partial t}(x,0) &= 2\omega\Big[\Big(\hat{b}_0 + (\hat{b}_{11} - \hat{b}_{12})\sin x + (\hat{b}_{22} + \hat{b}_{21})\cos 2x\Big)\sin\kappa x + \Big((\hat{b}_{11} + \hat{b}_{12})\cos x + (\hat{b}_{22} - \hat{b}_{21})\sin 2x\Big)\cos\kappa x\Big]. \end{aligned} \tag{78}$$

As is seen, these initial conditions correspond to the obtained analytical solution (50) for $\theta = 0$. Consequently, in accordance with this solution, at such initial conditions the beam should oscillate with the frequency ω. This allows for validating the obtained dispersion relations between frequency ω and wavenumber κ, as well as the obtained solution (50) itself.

Typical results of the numerical experiments are shown in Figure 8, where solid lines are values of the frequency ω obtained analytically (cf. Section 4), and filled circles represent numerical data for various values of the modulation amplitudes χ_I, χ_A and other parameters.

Figure 8(a) illustrates the pure effect of nonlinear curvature for $B = 0.5$, $\chi_A = 0.5$, $\chi_I = 0$, and Figure 8(b) the pure effect of nonlinear material for $\beta_n = 0.25$, $B = 0.5$, $\chi_A = \chi_I = 0.5$. The discrepancy between numerical and analytical values of the frequency ω is less than 0.3% for all values of κ, though in case (b) for $\kappa = 1$ it rises to 1%. Also additional (high-)frequency components are present in the beam response; these are due to the nonlinearity, and at the considered values of parameters do not exceed 3% of the total response amplitude; the larger the frequency ω, the more pronounced these components become.

Figure 8(c) illustrates the effect of initial pre-stretching of the *linear* beam for $\mu = 100$, $\eta = -0.002$, $\chi_A = 0$, $\chi_I = 0.5$. Here the discrepancy between numerical and analytical values of the frequency ω is even smaller, around 0.2%, and no additional frequency components are present in the beam response. Figure 8(d) represents mid-plane stretching nonlinearity for $\mu = 100$, $B = 0.06$, $\eta = 0.001$, $\chi_A = \chi_I = 0.5$.

It should be noted that Wolfram Mathematica, as well as other similar software packages, is not able to handle nonlinear partial integro-differential equations. Thus, when solving numerically the considered equation, the integral term was calculated using the obtained analytical solution. The

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resulting discrepancy between numerical and analytical values of the frequency ω is less than 0.5%, and the additional frequency components in the beam response do not exceed 5% of the total response amplitude. To validate the employed simplification, the integral term was calculated using the obtained numerical solution and compared with the analytical one; the resulting discrepancy between them was less than 0.6%.

Figure 8(e, f) illustrate the isolated effect of nonlinear inertia for $B = 0.4$, $\chi_A = 0.5$, $\chi_I = 0$. Here the discrepancy between numerical and analytical values of the frequency ω is again very small, around 0.2%. However, the additional (high-)frequency components in the beam response are about 10% for κ near ± 0.5 and ± 1, and the closer the wavenumber κ to the regions in which the wave motion is strongly nonlinear, the more pronounced these components become. Similar results were obtained for the case when all three sources of the nonlinearity are taken into account, see Figure 8(g, h) for $B = 0.4$ and (g) $\beta_n = 0.2$, $\chi_A = 0$, $\chi_I = 0.5$; (h) $\beta_n = 0.3$, $\chi_A = \chi_I = 0.5$. So, the magnitude of additional frequency components in the beam response is strongly dependent on the size of the nonlinearity.

In summary, good agreement between numerical and analytical results for all the considered cases can be noted, and thus the effects revealed in Section 4 are validated.

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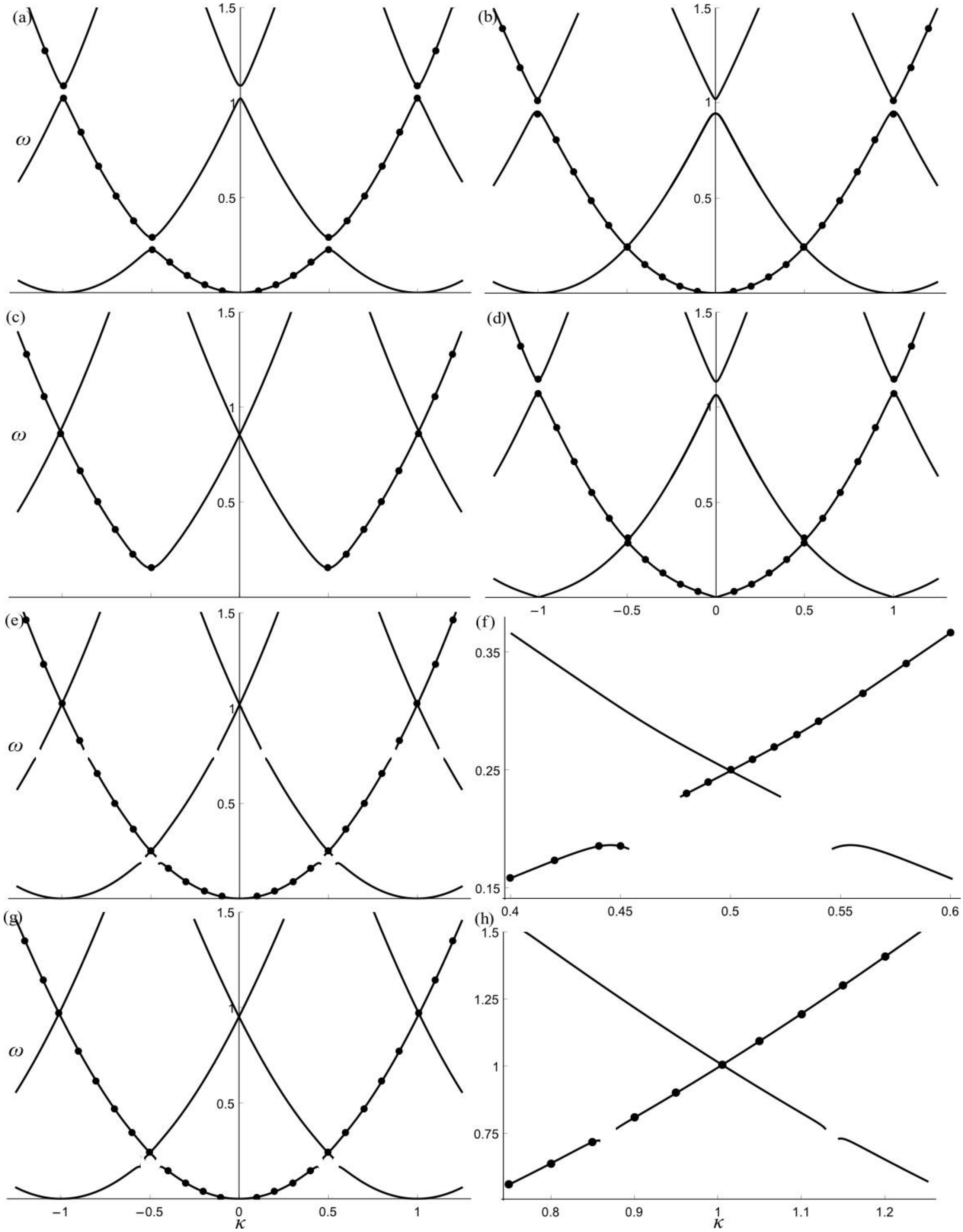

Figure 8. Dispersion relations $\omega(\kappa)$ illustrating (a) the isolated effect of nonlinear curvature $B=0.5$, $\chi_A=0.5$, $\chi_I=0$; (b) the isolated effect of nonlinear material $\beta_n=0.25$, $B=0.5$, $\chi_A=\chi_I=0.5$; (c) the isolated effect of initial pre-stretching of the linear beam $\mu=100$, $\eta=-0.002$, $\chi_A=0$, $\chi_I=0.5$; (d) the isolated effect of mid-plane stretching

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nonlinearity $\mu = 100$, $B = 0.06$, $\eta = 0.001$, $\chi_A = \chi_I = 0.5$; (e, f) the isolated effect of nonlinear inertia $B = 0.4$, $\chi_A = 0.5$, $\chi_I = 0$; (g, h) the combined effect of nonlinear inertia, nonlinear curvature and nonlinear material $B = 0.4$ and (g) $\beta_n = 0.2$, $\chi_A = 0$, $\chi_I = 0.5$; (h) $\beta_n = 0.3$, $\chi_A = \chi_I = 0.5$. Solid line: analytical results, filled circles: numerical data.

Conclusions

The effects of weak nonlinearity on the dispersion relation and frequency band-gaps of a periodic Bernoulli-Euler beam performing bending oscillations are analyzed. Two cases are considered: 1) large transverse deflections, where nonlinear (true) curvature, nonlinear material, and nonlinear inertia due to longitudinal motions of the beam are taken into account, and 2) mid-plane stretching nonlinearity. As a result several notable effects are revealed by the means of the Method of Varying Amplitudes. In particular, a shift of the band-gaps to a higher frequency due to nonlinear curvature is revealed, while the effect of nonlinear material is the opposite. The width of the band-gaps appears to be relatively insensitive to these nonlinearities. It is shown that initial pre-stretching of the beam considerably affects the dispersion relation: It is possible for new band-gaps to emerge and the band-gaps can be shifted to a higher or lower frequency, their width being considerably changed. The isolated effects of mid-plane stretching nonlinearity are similar to those of nonlinear curvature, though mid-plane stretching nonlinearity is pronounced already at much smaller beam deflections.

It is shown that of the four sources of nonlinearity considered, nonlinear inertia has the most substantial impact on the dispersion relation of a non-uniform beam with continuous modulations of cross-section parameters. It appears to remove all band-gaps, by making the wave motion either strongly nonlinear (for very small beam deflections) or weakly nonlinear and featuring no band-gaps (for larger beam deflections). The results obtained are validated by numerical simulation, and explanations of the revealed effects are suggested.

Experimental testing of the theoretical predictions could involve obtaining frequency responses for a beam with spatially periodic cross-section performing bending oscillations at clamped-clamped and clamped-free boundary conditions, and comparing with theoretical predictions as given in this paper for frequency band-gaps and propagation constants.

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